

Tannaka duality and convolution for duoidal categories

Thomas Booker* and Ross Street†

2000 Mathematics Subject Classification. 18D35; 18D10; 20J06

Key words and phrases. duoidal; duoid; bimonoid; tannaka; monoidal category.

Abstract

Given a horizontal monoid M in a duoidal category \mathcal{F} , we examine the relationship between bimonoid structures on M and monoidal structures on the category \mathcal{F}^{*M} of right M -modules which lift the vertical monoidal structure of \mathcal{F} . We obtain our result using a variant of the Tannaka adjunction. The approach taken utilizes hom-enriched categories rather than categories on which a monoidal category acts (“actegories”). The requirement of enrichment in \mathcal{F} itself demands the existence of some internal homs, leading to the consideration of convolution for duoidal categories. Proving that certain hom-functors are monoidal, and so take monoids to monoids, unifies classical convolution in algebra and Day convolution for categories. Hopf bimonoids are defined leading to a lifting of closed structures on \mathcal{F} to \mathcal{F}^{*M} . Warped monoidal structures permit the construction of new duoidal categories.

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*This author was supported by an Australian Postgraduate Award.

†This author gratefully acknowledges the support of an Australian Research Council Discovery Grant DP1094883.

1 Introduction

This paper initiates the development of a general theory of duoidal categories. In addition to providing the requisite definition of a duoidal \mathcal{V} -category, various “classical” concepts are reinterpreted and new notions put forth, including: produoidal \mathcal{V} -categories, convolution structures and duoidal cocompletion, enrichment in a duoidal \mathcal{V} -category, Tannaka duality, lifting closed structures to a category of representations (Hopf opmonoidal monads), and discovering new duoidal categories by “warping” the monoidal structure of another. Duoidal categories, some examples, and applications, have appeared in the Aguiar-Mahajan book [1] (under the name “2-monoidal categories”), in the recently published work of Batanin-Markl [2] and in a series of lectures by the second author [23]. Taken together with this paper, the vast potential of duoidal category theory is only now becoming apparent.

An encapsulated definition is that a duoidal \mathcal{V} -category \mathcal{F} is a pseudomonoid in the 2-category $\text{Mon}(\mathcal{V}\text{-Cat})$ of monoidal \mathcal{V} -categories, monoidal \mathcal{V} -functors and monoidal \mathcal{V} -natural transformations. Since $\text{Mon}(\mathcal{V}\text{-Cat})$ is equivalently the category of pseudomonoids in $\mathcal{V}\text{-Cat}$ we are motivated to call a pseudomonoid in a monoidal bicategory a *monoidale* (i.e. a monoidal object). Thus a duoidal \mathcal{V} -category is an object of $\mathcal{V}\text{-Cat}$ equipped with two monoidal structures, one called horizontal and the other called vertical, such that one is monoidal with respect to the other. Calling such an object a *duoidale* encourages one to consider duoidales in other monoidal bicategories, in particular $\mathcal{M} = \mathcal{V}\text{-Mod}$. By giving a canonical monoidal structure on the $\mathcal{V} = \mathcal{M}(\mathcal{I}, \mathcal{I})$ valued-hom for any left unit closed monoidal bicategory \mathcal{M} (see Section 2), we see that a duoidale in $\mathcal{M} = \mathcal{V}\text{-Mod}$ is precisely the notion of promonoidal category lifted to the duoidal setting, that is, a produoidal \mathcal{V} -category.

A study of duoidal cocompletion (in light of the produoidal \mathcal{V} -category material) leads to Section 5 where we consider enrichment in a duoidal \mathcal{V} -category base. We observe that if \mathcal{F} is a duoidal \mathcal{V} -category then the vertical monoidal structure \circ lifts to give a monoidal structure on $\mathcal{F}_h\text{-Cat}$. If \mathcal{F} is then a horizontally left closed duoidal \mathcal{V} -category then \mathcal{F} is in fact a monoidale $(\mathcal{F}_h, \hat{\circ}, \ulcorner \mathbf{1} \urcorner)$ in $\mathcal{F}_h\text{-Cat}$ with multiplication $\hat{\circ} : \mathcal{F}_h \circ \mathcal{F}_h \rightarrow \mathcal{F}_h$ defined using the evaluation of homs. That is, \mathcal{F}_h is an \mathcal{F}_h -category.

Section 6 revisits the Tannaka adjunction as it pertains to duoidal \mathcal{V} -categories. We write $\mathcal{F}_h\text{-Cat} \downarrow^{\text{ps}} \mathcal{F}_h$ for the 2-category $\mathcal{F}_h\text{-Cat} \downarrow \mathcal{F}_h$ restricted to having 1-cells those triangles that commute up to an isomorphism. Post composition with the monoidale multiplication $\hat{\circ}$ yields a tensor product $\bar{\circ}$ on $\mathcal{F}_h\text{-Cat} \downarrow^{\text{ps}} \mathcal{F}_h$ and we write $\mathcal{F}\text{-Cat} \downarrow^{\text{ps}} \mathcal{F}$ for this monoidal 2-category. Let \mathcal{F}^{*M} be the \mathcal{F}_h -category of Eilenberg-Moore algebras for the monad $- * M$. There is a monoidal functor $\text{mod} : (\text{Mon } \mathcal{F})^{\text{op}} \rightarrow \mathcal{F}\text{-Cat} \downarrow^{\text{ps}} \mathcal{F}$ defined by taking a monoid M to the object $U_M : \mathcal{F}^{*M} \rightarrow \mathcal{F}_h$. Here $\text{Mon } \mathcal{F}$ is only being considered as a monoidal category, not a 2-category. Representable objects of $\mathcal{F}\text{-Cat} \downarrow^{\text{ps}} \mathcal{F}$ are closed under the monoidal structure $\bar{\circ}$ which motivates restricting to $\mathcal{F}\text{-Cat} \downarrow_{\text{rep}}^{\text{ps}} \mathcal{F}$. Since representable functors are “tractable” and the functor $\text{end} : \mathcal{F}\text{-Cat} \downarrow_{\text{rep}}^{\text{ps}} \mathcal{F} \rightarrow \text{Mon } \mathcal{F}$ is strong monoidal we have the biadjunction

$$(\text{Bimon } \mathcal{F}_h)^{\text{op}} \begin{array}{c} \xleftarrow{\text{end}} \\ \perp \\ \xrightarrow{\text{mod}} \end{array} \text{Mon}_{\text{ps}}(\mathcal{F}\text{-Cat} \downarrow_{\text{rep}}^{\text{ps}} \mathcal{F})$$

giving the correspondence between bimonoid structures on M and isomorphism classes of monoidal structures on \mathcal{F}^{*M} such that the underlying functor is strong monoidal into the vertical structure on \mathcal{F} . The non-duoidal version of this result is attributed to Bodo Pareigis (see [3], [4] and [5]).

The notion of a Hopf opmonoidal monad is found in the paper of Bruguières-Lack-Virelizier [6]. We adapt their work to the duoidal setting in order to lift closed structures on the monoidale (monoidal \mathcal{F}_h -category) $(\mathcal{F}, \hat{\circ}, \lceil \mathbf{1} \rceil)$ to the \mathcal{F}_h -category of right modules \mathcal{F}^{*M} for a bimonoid M . In particular, Proposition 22 says that a monoidal \mathcal{F}_h -category $(\mathcal{F}, \hat{\circ}, \lceil \mathbf{1} \rceil)$ is closed if and only if \mathcal{F}_v is a closed monoidal \mathcal{V} -category and there exists \mathcal{V} -natural isomorphisms $X \circ (W * Y) \cong W * (X \circ Y) \cong (W * X) \circ Y$. In light of \mathcal{F} being a duoidal \mathcal{V} -category, Proposition 23 gives a refinement of this result which taken together with Proposition 22 yields two isomorphisms

$$X * (J \circ Y) \cong X \circ Y \cong Y * (X \circ J)$$

and

$$Y \circ (W * \mathbf{1}) \cong W * Y \cong (W * \mathbf{1}) \circ Y.$$

This result implies that in order to know \circ we only need to know $*$ and $J \circ -$ or $- \circ J$. Similarly to know $*$ we need only know \circ and $\mathbf{1} * -$ or $- * \mathbf{1}$. This extreme form of interpolation motivates the material of Section 8.

We would like a way to generate new duoidal categories. One possible method presented here is the notion of a *warped monoidal structure*. In its simplest presentation, a warping for a monoidal category $\mathcal{A} = (\mathcal{A}, \otimes)$ is a perturbation of \mathcal{A} 's tensor product by a “suitable” endo-functor $T : \mathcal{A} \rightarrow \mathcal{A}$ such that the new tensor product is defined by

$$A \square B = TA \otimes B.$$

We lift this definition to the level of a monoidale A in a monoidal bicategory \mathcal{M} . Proposition 26 observes that a warping for a monoidale determines another monoidale structure on A . If \mathcal{F} is a duoidal \mathcal{V} -category satisfying the right-hand side of the second isomorphism above then a vertical warping of \mathcal{F} by $T = - * \mathbf{1}$ recovers \mathcal{F}_h . This is precisely a warping of the monoidale \mathcal{F}_v in $\mathcal{M} = \mathcal{V}\text{-Cat}$. The last example given generates a duoidal category by warping the monoidal structure of any lax braided monoidal category viewed as a duoidal category with $* = \circ = \otimes$ and $\gamma = 1 \otimes c \otimes 1$.

2 The monoidality of hom

Let (\mathcal{V}, \otimes) be a symmetric closed complete and cocomplete monoidal category. Recall from [17] that a \mathcal{V} -natural transformation θ between \mathcal{V} -functors $T, S : \mathcal{A} \rightarrow \mathcal{X}$ consists of a \mathcal{V} -natural family

$$\theta_A : TA \longrightarrow SA, A \in \mathcal{A},$$

such that the diagram

$$\begin{array}{ccc} \mathcal{A}(A, B) & \xrightarrow{T} & \mathcal{X}(TA, TB) \\ S \downarrow & & \downarrow \mathcal{X}(1, \theta_B) \\ \mathcal{X}(SA, SB) & \xrightarrow{\mathcal{X}(\theta_A, 1)} & \mathcal{X}(TA, SB) \end{array}$$

commutes in the base category \mathcal{V} .

If (\mathcal{C}, \boxtimes) is a monoidal \mathcal{V} -category with tensor product \boxtimes then the associativity isomorphisms $a_{A,B,C} : (A \boxtimes B) \boxtimes C \rightarrow A \boxtimes (B \boxtimes C)$ are necessarily a \mathcal{V} -natural family, which amounts to the commutativity of the diagram

$$\begin{array}{ccc} (\mathcal{C}(A, A') \otimes \mathcal{C}(B, B')) \otimes \mathcal{C}(C, C') & \xrightarrow{\boxtimes(\boxtimes \otimes 1)} & \mathcal{C}((A \boxtimes B) \boxtimes C, (A' \boxtimes B') \boxtimes C') \\ \cong \downarrow & & \downarrow \mathcal{C}(1, a_{A', B', C'}) \\ \mathcal{C}(A, A') \otimes (\mathcal{C}(B, B') \otimes \mathcal{C}(C, C')) & \xrightarrow{Nat_a} & \\ \boxtimes(1 \otimes \boxtimes) \downarrow & & \downarrow \\ \mathcal{C}(A \boxtimes (B \boxtimes C), A' \boxtimes (B' \boxtimes C')) & \xrightarrow{\mathcal{C}(a_{A, B, C}, 1)} & \mathcal{C}((A \boxtimes B) \boxtimes C, A' \boxtimes (B' \boxtimes C')) \end{array}$$

Similarly the \mathcal{V} -naturality of the unit isomorphisms

$$\ell_A : I \boxtimes A \longrightarrow A \quad \text{and} \quad r_A : A \boxtimes I \longrightarrow A$$

amounts to the commutativity of

$$\begin{array}{ccc} \mathcal{C}(A, A') \xrightarrow{I \boxtimes -} \mathcal{C}(I \boxtimes A, I \boxtimes A') & & \mathcal{C}(A, A') \xrightarrow{- \boxtimes I} \mathcal{C}(A \boxtimes I, A' \boxtimes I) \\ \parallel & \downarrow \mathcal{C}(1, \ell_{A'}) & \parallel \\ \mathcal{C}(A, A') \xrightarrow{\mathcal{C}(\ell_A, 1)} \mathcal{C}(I \boxtimes A, A') & & \mathcal{C}(A, A') \xrightarrow{\mathcal{C}(r_A, 1)} \mathcal{C}(A \boxtimes I, A') \end{array}$$

Proposition 1 *If (\mathcal{C}, \boxtimes) is a monoidal \mathcal{V} -category then the \mathcal{V} -functor*

$$\mathcal{C}(-, -) : \mathcal{C}^{\text{op}} \otimes \mathcal{C} \longrightarrow \mathcal{V}$$

is equipped with a canonical monoidal structure.

Proof For $\mathcal{C}(-, -)$ to be monoidal we require the morphisms

$$\boxtimes : \mathcal{C}(W, X) \otimes \mathcal{C}(Y, Z) \longrightarrow \mathcal{C}(W \boxtimes Y, X \boxtimes Z)$$

and

$$j_I : I \longrightarrow \mathcal{C}(I, I)$$

to satisfy the axioms

$$\begin{array}{ccc} (\mathcal{C}(U, V) \otimes \mathcal{C}(W, X)) \otimes \mathcal{C}(Y, Z) & \xrightarrow{\boxtimes 1} & \mathcal{C}(U \boxtimes W, V \boxtimes X) \otimes \mathcal{C}(Y, Z) \\ \cong \downarrow & & \downarrow \boxtimes \\ \mathcal{C}(U, V) \otimes (\mathcal{C}(W, X) \otimes \mathcal{C}(Y, Z)) & & \mathcal{C}((U \boxtimes W) \boxtimes Y, (V \boxtimes X) \boxtimes Z) \\ 1 \otimes \boxtimes \downarrow & & \downarrow \mathcal{C}(a_{U, W, Y}^{-1}, a_{V, X, Z}) \\ \mathcal{C}(U, V) \otimes \mathcal{C}(W \boxtimes Y, X \boxtimes Z) & \xrightarrow{\boxtimes} & \mathcal{C}(U \boxtimes (W \boxtimes Y), V \boxtimes (X \boxtimes Z)) \end{array}$$

and

$$\begin{array}{ccc} \mathcal{C}(I, I) \otimes \mathcal{C}(Y, Z) & \xrightarrow{\boxtimes} & \mathcal{C}(I \boxtimes Y, I \boxtimes Z) \\ j_I \otimes 1 \uparrow & & \downarrow \mathcal{C}(\ell_Y^{-1}, \ell_Z) \\ I \otimes \mathcal{C}(Y, Z) & \xrightarrow{\ell} & \mathcal{C}(Y, Z) \end{array} \quad \begin{array}{ccc} \mathcal{C}(W, X) \otimes I & \xrightarrow{r} & \mathcal{C}(W, X) \\ 1 \otimes j_I \downarrow & & \uparrow \mathcal{C}(r_W^{-1}, r_X) \\ \mathcal{C}(W, X) \otimes \mathcal{C}(I, I) & \xrightarrow{\boxtimes} & \mathcal{C}(W \boxtimes I, X \boxtimes I) \end{array}$$

These diagrams are simply reorganizations of the diagrams Nat_a , Nat_ℓ , and Nat_r above. \blacksquare

Corollary 2 *If \mathcal{C} is a comonoid and A is a monoid in the monoidal \mathcal{V} -category \mathcal{C} then $\mathcal{C}(\mathcal{C}, A)$ is canonically a monoid in \mathcal{V} .*

Proof We observe that monoidal \mathcal{V} -functors take monoids to monoids and (\mathcal{C}, A) is a monoid in $\mathcal{C}^{\text{op}} \otimes \mathcal{C}$. \blacksquare

Proposition 3 *If \mathcal{C} is a braided monoidal \mathcal{V} -category then*

$$\mathcal{C}(-, -) : \mathcal{C}^{\text{op}} \otimes \mathcal{C} \longrightarrow \mathcal{V}$$

is a braided monoidal \mathcal{V} -functor.

Proof Let $c_{X, Y} : X \boxtimes Y \longrightarrow Y \boxtimes X$ denote the braiding on \mathcal{C} . The requirement of \mathcal{V} -naturality for this family of isomorphisms amounts precisely to the commutativity of

$$\begin{array}{ccc} \mathcal{C}(W, X) \otimes \mathcal{C}(Y, Z) & \xrightarrow{\boxtimes} & \mathcal{C}(W \boxtimes Y, X \boxtimes Z) \\ \cong \downarrow & & \downarrow \mathcal{C}(c^{-1}, c) \\ \mathcal{C}(Y, Z) \otimes \mathcal{C}(W, X) & \xrightarrow{\boxtimes} & \mathcal{C}(Y \boxtimes W, Z \boxtimes X) \end{array}$$

which is exactly the braiding condition for the monoidal functor $\mathcal{C}(-, -)$ of Proposition 1. \blacksquare

We now give a spiritual successor to the above by moving to the level of monoidal bicategories.

Proposition 4 *If \mathcal{M} is a monoidal bicategory then the pseudofunctor*

$$\mathcal{M}(-, -) : \mathcal{M}^{\text{op}} \times \mathcal{M} \longrightarrow \text{Cat}$$

is equipped with a canonical monoidal structure.

Proof We avail ourselves of the coherence theorem of [13] by assuming that \mathcal{M} is a Gray monoid (see [10]). The definition of a monoidal pseudofunctor (called a “weak monoidal homomorphism”) between Gray monoids is defined on pages 102 and 104 of [10]. Admittedly Cat is not a Gray monoid, but the adjustment to compensate for this is not too challenging.

In the notation of [10], the pseudonatural transformation χ is defined at objects to be the functor

$$\otimes : \mathcal{M}(A, A') \times \mathcal{M}(B, B') \longrightarrow \mathcal{M}(A \otimes B, A' \otimes B')$$

and at the morphisms to be the isomorphism

$$\begin{array}{ccc} \mathcal{M}(A, A') \times \mathcal{M}(B, B') & \xrightarrow{\otimes} & \mathcal{M}(A \otimes B, A' \otimes B') \\ \mathcal{M}(f, f') \times \mathcal{M}(g, g') \downarrow & \cong & \downarrow \mathcal{M}(f, g) \times \mathcal{M}(f', g') \\ \mathcal{M}(C, C') \times \mathcal{M}(D, D') & \xrightarrow{\otimes} & \mathcal{M}(C \otimes D, C' \otimes D') \end{array}$$

whose component

$$(f'uf) \otimes (g'vg) \cong (f' \otimes g')(u \otimes v)(f \otimes g)$$

at $(u, v) \in \mathcal{M}(A, A') \times \mathcal{M}(B, B')$ is the canonical isomorphism associated with the pseudofunctor $\otimes : \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$ (see the top of page 102 of [10]). For ι , we have the functor $1 \longrightarrow \mathcal{M}(I, I)$ which picks out 1_I . For ω , we have the natural isomorphism

$$\begin{array}{ccc} \mathcal{M}(A, A') \times \mathcal{M}(B, B') \times \mathcal{M}(C, C') & \xrightarrow{\otimes \times 1} & \mathcal{M}(A \otimes B, A' \otimes B') \times \mathcal{M}(C, C') \\ 1 \times \otimes \downarrow & \omega \Downarrow & \downarrow \otimes \\ \mathcal{M}(A, A') \times \mathcal{M}(B \otimes C, B' \otimes C') & \xrightarrow{\otimes} & \mathcal{M}(A \otimes B \otimes C, A' \otimes B' \otimes C') \end{array}$$

whose component at (u, v, w) is the canonical isomorphism

$$(u \otimes v) \otimes w \cong u \otimes (v \otimes w)$$

associated with $\otimes : \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$. For ξ and κ , we have the natural isomorphisms

$$\begin{array}{ccc} & \mathcal{M}(A, A') \times \mathcal{M}(I, I) & \\ 1 \times \lceil 1_I \rceil \nearrow & \cong \Downarrow & \searrow \otimes \\ \mathcal{M}(A, A') & \xrightarrow{1} & \mathcal{M}(A, A') \end{array}$$

and

$$\begin{array}{ccc}
& \mathcal{M}(I, I) \times \mathcal{M}(A, A') & \\
\nearrow \lceil 1_I \rceil \times 1 & \cong \Downarrow & \searrow \otimes \\
\mathcal{M}(A, A') & \xrightarrow{1} & \mathcal{M}(A, A')
\end{array}$$

with canonical components

$$u \otimes 1_I \cong u \quad \text{and} \quad 1_I \otimes u \cong u.$$

The two required axioms are then a consequence of the coherence conditions for pseudofunctors in the case of $\otimes : \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}$. \blacksquare

Corollary 5 ([10]; page 110, Proposition 4) *If A is a pseudomonoid and C is a pseudocomonoid in a monoidal bicategory \mathcal{M} then the category $\mathcal{M}(C, A)$ is equipped with a canonical monoidal structure.*

Proposition 6 *If \mathcal{M} is a braided monoidal bicategory then*

$$\mathcal{M}(-, -) : \mathcal{M}^{\text{op}} \times \mathcal{M} \longrightarrow \text{Cat}$$

is a braided monoidal pseudofunctor.

Proof The required data of page 122, Definition 14 in [10] is provided by the invertible modification

$$\begin{array}{ccc}
\mathcal{M}(A, A') \times \mathcal{M}(B, B') & \xrightarrow{\cong} & \mathcal{M}(B, B') \times \mathcal{M}(A, A') \\
\downarrow \otimes & \Downarrow \cong & \downarrow \otimes \\
\mathcal{M}(A \otimes B, A' \otimes B') & \xrightarrow{\mathcal{M}(\rho^{-1}, \rho)} & \mathcal{M}(B \otimes A, B' \otimes A')
\end{array}$$

whose component at (u, v) is

$$\begin{array}{ccccc}
B \otimes A & \xrightarrow{\rho} & A \otimes B & \xrightarrow{u \otimes v} & A' \otimes B' \\
& \searrow \cong & \downarrow \rho & \cong \rho_{u,v} & \downarrow \rho \\
& & B \otimes A & \xrightarrow{v \otimes u} & B' \otimes A'
\end{array}$$

1

\blacksquare

What we really want is a presentation of these results lifted to the level of enriched monoidal bicategories.

Suppose \mathcal{M} is a monoidal bicategory. Put $\mathcal{V} = \mathcal{M}(I, I)$, regarding it as a monoidal category under composition \circ . There is another “multiplication” on \mathcal{V} defined by the composite

$$\mathcal{M}(I, I) \times \mathcal{M}(I, I) \xrightarrow{\otimes} \mathcal{M}(I \otimes I, I \otimes I) \cong \mathcal{M}(I, I)$$

with the same unit 1_I as \circ . By Proposition 5.3 of [16], a braiding is obtained on \mathcal{V} .

Furthermore, each hom category $\mathcal{M}(X, Y)$ has an action

$$\mathcal{M}(I, I) \times \mathcal{M}(X, Y) \xrightarrow{\otimes} \mathcal{M}(I \otimes X, I \otimes Y) \simeq \mathcal{M}(X, Y)$$

by \mathcal{V} which we abusively write as

$$(v, m) \longmapsto v \otimes m .$$

We call \mathcal{M} *left unit closed* when each functor

$$- \otimes m : \mathcal{V} \longrightarrow \mathcal{M}(X, Y)$$

has a right adjoint

$$[m, -] : \mathcal{M}(X, Y) \longrightarrow \mathcal{V} .$$

That is, we have a natural isomorphism

$$\mathcal{M}(X, Y)(v \otimes m, n) \cong \mathcal{V}(v, [m, n]) .$$

In particular, this implies \mathcal{V} is a left closed monoidal category and that each hom category $\mathcal{M}(X, Y)$ is \mathcal{V} -enriched with \mathcal{V} -valued hom defined by $[m, n]$. Furthermore, since \mathcal{V} is braided, the 2-category $\mathcal{V}\text{-Cat}$ of \mathcal{V} -categories, \mathcal{V} -functors and \mathcal{V} -natural transformations is monoidal; see Remark 5.2 of [16].

Proposition 7 *If the monoidal bicategory \mathcal{M} is left unit closed then the monoidal pseudofunctor of Proposition 4 lifts to a monoidal pseudofunctor*

$$\mathcal{M}(-, -) : \mathcal{M}^{\text{op}} \times \mathcal{M} \longrightarrow \mathcal{V}\text{-Cat}$$

where $\mathcal{V} = \mathcal{M}(I, I)$ as above.

Proof We use the fact that, for tensored \mathcal{V} -categories \mathcal{A} and \mathcal{B} , enrichment of a functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ to a \mathcal{V} -functor can be expressed in terms of a lax action morphism structure

$$\bar{\chi}_{V, A} : V \otimes FA \longrightarrow F(V \otimes A)$$

for $V \in \mathcal{V}$, $A \in \mathcal{A}$. Given such \mathcal{V} -functors $F, G : \mathcal{A} \longrightarrow \mathcal{B}$, a family of morphisms

$$\theta_A : FA \longrightarrow GA$$

is \mathcal{V} -natural if and only if the diagrams

$$\begin{array}{ccc} V \otimes FA & \xrightarrow{\bar{\chi}_{V, A}} & F(V \otimes A) \\ 1 \otimes \theta_A \downarrow & & \downarrow \theta_{V \otimes A} \\ V \otimes GA & \xrightarrow{\bar{\chi}_{V, A}} & G(V \otimes A) \end{array}$$

commute. Therefore, to see that the functors

$$\mathcal{M}(f, g) : \mathcal{M}(X, Y) \longrightarrow \mathcal{M}(X', Y') ,$$

for $f : X' \longrightarrow X$ and $g : Y \longrightarrow Y'$, are \mathcal{V} -enriched, we require 2-cells

$$v \otimes (g \circ m \circ f) \longrightarrow g \circ (v \otimes m) \circ f$$

which constitute a lax action morphism. As in the proof of Proposition 4, we assume that \mathcal{M} is a Gray monoid where we can take these 2-cells to be the canonical isomorphisms. It is then immediate that the 2-cells $\sigma : f \Longrightarrow f'$ and $\tau : g \Longrightarrow g'$ induce \mathcal{V} -natural transformations $\mathcal{M}(\sigma, \tau) : \mathcal{M}(f, g) \Longrightarrow \mathcal{M}(f', g')$.

For the monoidal structure on $\mathcal{M}(-, -)$, we need to see that the effect of the tensor of \mathcal{M} on homs defines a \mathcal{V} -functor

$$\otimes : \mathcal{M}(A, A') \otimes \mathcal{M}(B, B') \longrightarrow \mathcal{M}(A \otimes B, A' \otimes B') .$$

Again we make use of the coherent isomorphisms; in this case they are

$$v \otimes (m \otimes n) \cong (v \otimes m) \otimes (v \otimes n)$$

for $v : I \longrightarrow I$, $m : A \longrightarrow A'$, $n : B \longrightarrow B'$. It is clear that ι can be regarded as a \mathcal{V} -functor $\iota : \mathcal{I} \longrightarrow \mathcal{M}(I, I)$. The \mathcal{V} -naturality of all the 2-cells involved in the monoidal structure on $\mathcal{M}(-, -)$ now follows automatically from the naturality of the Gray monoid constraints. ■

Proposition 8 *In the situation of Proposition 7, if \mathcal{M} is also symmetric then so is $\mathcal{M}(-, -)$.*

Proof If \mathcal{M} is symmetric, so too is $\mathcal{V} = \mathcal{M}(I, I)$. Consequently, $\mathcal{V}\text{-Cat}$ is also symmetric. Referring to the proof of Proposition 6, we see that the techniques of the proof of Proposition 7 apply. ■

Example Let \mathcal{V} be any braided monoidal category which is closed complete and cocomplete. Put $\mathcal{M} = \mathcal{V}\text{-Mod}$, the bicategory of \mathcal{V} -categories, \mathcal{V} -modules (i.e. \mathcal{V} -distributors or equivalently \mathcal{V} -profunctors), and \mathcal{V} -module morphisms. This \mathcal{M} is a well-known example of a monoidal bicategory 10. We can easily identify \mathcal{V} with $\mathcal{V}\text{-Mod}(\mathcal{I}, \mathcal{I})$ and the action on $\mathcal{M}(\mathcal{A}, \mathcal{X})$ with the functor

$$\mathcal{V} \times \mathcal{V}\text{-Mod}(\mathcal{A}, \mathcal{X}) \longrightarrow \mathcal{V}\text{-Mod}(\mathcal{A}, \mathcal{X})$$

given by the mapping

$$(V, M) \longmapsto V \otimes M$$

defined by $(V \otimes M)(X, A) = V \otimes M(X, A)$ with left module action

$$\mathcal{A}(A, B) \otimes V \otimes M(X, A) \xrightarrow[\cong]{c \otimes 1} V \otimes \mathcal{A}(A, B) \otimes M(X, A) \xrightarrow{1 \otimes \text{act}_\ell} V \otimes M(X, B)$$

and right module action

$$V \otimes M(X, A) \otimes \mathcal{X}(Y, X) \xrightarrow{1 \otimes r} V \otimes M(Y, A) ,$$

where c is the braiding of \mathcal{V} and we have ignored associativity isomorphisms. To see that $\mathcal{M} = \mathcal{V}\text{-Mod}$ is left unit closed we easily identify $[M, N] \in \mathcal{V}$ for $M, N \in$

$\mathcal{V}\text{-Mod}(\mathcal{A}, \mathcal{X})$ with the usual \mathcal{V} -valued hom for the \mathcal{V} -category $[\mathcal{X}^{\text{op}} \otimes \mathcal{A}, \mathcal{V}]$; namely,

$$[M, N] = \int_{X, A} [M(X, A), N(X, A)],$$

the “object of \mathcal{V} -natural transformations”. Therefore, in this case, Proposition 7 is about the pseudofunctor

$$\mathcal{V}\text{-Mod}^{\text{op}} \times \mathcal{V}\text{-Mod} \longrightarrow \mathcal{V}\text{-Cat},$$

given by the mapping

$$(\mathcal{A}, \mathcal{X}) \longmapsto [\mathcal{X}^{\text{op}} \otimes \mathcal{A}, \mathcal{V}],$$

asserting monoidality. When \mathcal{V} is symmetric, Proposition 8 assures us the pseudofunctor is also symmetric.

Remark There is presumably a more general setting encompassing the results of this section. For a monoidal bicategory \mathcal{K} , it is possible to define a notion of \mathcal{K} -*bicategory* \mathcal{M} by which we mean that the homs $\mathcal{M}(X, Y)$ are objects of \mathcal{K} . For Proposition 1 we would take \mathcal{K} to be \mathcal{V} as a locally discrete bicategory and \mathcal{M} to be \mathcal{C} . For Proposition 4, \mathcal{K} would be Cat . For Proposition 7, \mathcal{K} would be $\mathcal{V}\text{-Cat}$. Then, as in these cases, we would require \mathcal{K} to be braided in order to define the *tensor product of \mathcal{K} -bicategories* and so *monoidal \mathcal{K} -bicategories*. With all this properly defined, we expect

$$\mathcal{M}(-, -) : \mathcal{M}^{\text{op}} \otimes \mathcal{M} \longrightarrow \mathcal{K}$$

to be a monoidal \mathcal{K} -pseudofunctor.

3 Duoidal \mathcal{V} -categories

Throughout \mathcal{V} is a symmetric monoidal closed, complete and cocomplete category. The following definition agrees with that of Batanin and Markl in [2] and, under the name 2-monoidal category, Aguiar and Mahajan in [1].

Definition 1 A *duoidal structure* on a \mathcal{V} -category \mathcal{F} consists of two \mathcal{V} -monoidal structures

$$* : \mathcal{F} \otimes \mathcal{F} \longrightarrow \mathcal{F}, \quad \lceil J \rceil : \mathbf{1} \longrightarrow \mathcal{F}, \quad (3.1)$$

$$\circ : \mathcal{F} \otimes \mathcal{F} \longrightarrow \mathcal{F}, \quad \lceil 1 \rceil : \mathbf{1} \longrightarrow \mathcal{F}, \quad (3.2)$$

such that either of the following equivalent conditions holds:

- (i) the \mathcal{V} -functors \circ and $\lceil 1 \rceil$ of (3.2) and their coherence isomorphisms are monoidal with respect to the monoidal \mathcal{V} -category \mathcal{F}_h of (3.1).
- (ii) the \mathcal{V} -functors $*$ and $\lceil J \rceil$ of (3.1) and their coherence isomorphisms are opmonoidal with respect to the monoidal \mathcal{V} -category \mathcal{F}_v of (3.2).

We call the monoidal \mathcal{V} -category \mathcal{F}_h of (3.1) *horizontal* and the monoidal \mathcal{V} -category \mathcal{F}_v of (3.2) *vertical*; this terminology comes from an example of derivation schemes due to [2] (also see [23]).

The extra elements of structure involved in (i) and (ii) are a \mathcal{V} -natural middle-of-four interchange transformation

$$\gamma : (A \circ B) * (C \circ D) \longrightarrow (A * C) \circ (B * D) ,$$

and maps

$$\mathbf{1} * \mathbf{1} \xrightarrow{\mu} \mathbf{1} \xleftarrow{\tau} J \xrightarrow{\delta} J \circ J$$

such that the diagrams

$$\begin{array}{ccc} ((A \circ B) * (C \circ D)) * (E \circ F) & \xrightarrow{\cong} & (A \circ B) * ((C \circ D) * (E \circ F)) \\ \gamma * 1 \downarrow & & \downarrow 1 * \gamma \\ ((A * C) \circ (B * D)) * (E \circ F) & & (A \circ B) * ((C * E) \circ (D * F)) \\ \gamma \downarrow & & \downarrow \gamma \\ ((A * C) * E) \circ ((B * D) * F) & \xrightarrow{\cong} & (A * (C * E)) \circ (B * (D * F)) \end{array} \quad (3.3)$$

$$\begin{array}{ccc} ((A \circ B) \circ C) * ((D \circ E) \circ F) & \xrightarrow{\cong} & (A \circ (B \circ C)) * (D \circ (E \circ F)) \\ \gamma \downarrow & & \downarrow \gamma \\ ((A \circ B) * (D \circ E)) \circ (C * F) & & (A * D) \circ ((B \circ C) * (E \circ F)) \\ \gamma \circ 1 \downarrow & & \downarrow 1 \circ \gamma \\ ((A * D) \circ (B * E)) \circ (C * F) & \xrightarrow{\cong} & (A * D) \circ ((B * E) \circ (C * F)) \end{array} \quad (3.4)$$

and

$$\begin{array}{ccc} J * (A \circ B) \xrightarrow{\delta * 1} (J \circ J) * (A \circ B) & (A \circ B) * J \xrightarrow{1 * \delta} (A \circ B) * (J \circ J) & (3.5) \\ \cong \uparrow & \downarrow \gamma & \\ A \circ B \xrightarrow{\cong} (J * A) \circ (J * B) & & A \circ B \xrightarrow{\cong} (A * J) \circ (B * J) \end{array}$$

$$\begin{array}{ccc} \mathbf{1} \circ (A * B) \xleftarrow{\mu \circ 1} (\mathbf{1} * \mathbf{1}) \circ (A * B) & (A * B) \circ \mathbf{1} \xleftarrow{1 \circ \mu} (A * B) \circ (\mathbf{1} * \mathbf{1}) & (3.6) \\ \cong \uparrow & \uparrow \gamma & \\ A * B \xrightarrow{\cong} (\mathbf{1} \circ A) * (\mathbf{1} \circ B) & & A * B \xrightarrow{\cong} (A \circ \mathbf{1}) * (B \circ \mathbf{1}) \end{array}$$

commute, together with the requirement that $(\mathbf{1}, \mu, \tau)$ is a monoid in \mathcal{F}_h and (J, δ, τ) is a comonoid in \mathcal{F}_v .

Example A braided monoidal category \mathcal{C} with braid isomorphism $c : A \otimes B \cong B \otimes A$ is an example of a duoidal category with $\otimes = * = \circ$ and γ , determined by $1_A \otimes c \otimes 1_D$ and re-bracketing, invertible.

Example Let \mathcal{C} be a monoidal \mathcal{V} -category. An important example is the \mathcal{V} -category $\mathcal{F} = [\mathcal{C}^{\text{op}} \otimes \mathcal{C}, \mathcal{V}]$ of \mathcal{V} -modules $\mathcal{C} \dashrightarrow \mathcal{C}$ and \mathcal{V} -module homomorphisms. We see that \mathcal{F} becomes a duoidal \mathcal{V} -category with $*$ the convolution tensor product for $\mathcal{C}^{\text{op}} \otimes \mathcal{C}$ and \circ the tensor product “over \mathcal{C} ”. This example can be found in [23].

Definition 2 A *duoidal functor* $F : \mathcal{F} \rightarrow \mathcal{F}'$ is a functor F that is equipped with monoidal structures $\mathcal{F}_h \rightarrow \mathcal{F}'_h$ and $\mathcal{F}_v \rightarrow \mathcal{F}'_v$ which are compatible with the duoidal data γ, μ, δ , and τ .

Definition 3 A *bimonoidal functor* $T : \mathcal{F} \rightarrow \mathcal{F}'$ is a functor F that is equipped with a monoidal structure $\mathcal{F}_h \rightarrow \mathcal{F}'_h$ and an opmonoidal structure $\mathcal{F}_v \rightarrow \mathcal{F}'_v$ both of which are compatible with the duoidal data γ, μ, δ , and τ .

Definition 4 A *bimonoid* A in a duoidal category \mathcal{F} is a bimonoidal functor $\lceil A \rceil : \mathbf{1} \rightarrow \mathcal{F}$. That is, it is an object A equipped with the structure of a monoid for $*$ and a comonoid for \circ , compatible via the axioms

$$\begin{array}{ccc} A * A & \xrightarrow{\mu} & A \xrightarrow{\delta} A \circ A \\ \delta * \delta \downarrow & & \uparrow \mu \circ \mu \\ (A \circ A) * (A \circ A) & \xrightarrow{\gamma} & (A * A) \circ (A * A) \end{array} \quad (3.7)$$

$$\begin{array}{ccc} A * A & \xrightarrow{\mu} & A \\ \epsilon * \epsilon \downarrow & = & \downarrow \epsilon \\ \mathbf{1} * \mathbf{1} & \xrightarrow{\mu} & \mathbf{1} \end{array} \quad \begin{array}{ccc} J \circ J & \xleftarrow{\delta} & J \\ \eta \circ \eta \downarrow & = & \downarrow \eta \\ A \circ A & \xleftarrow{\delta} & A \end{array} \quad (3.8)$$

$$\begin{array}{ccc} J & & \\ \tau \downarrow & \searrow \eta & \\ & = & A \\ & \swarrow \epsilon & \\ \mathbf{1} & & \end{array} \quad (3.9)$$

These are a lifting of the usual axioms for a bimonoid in a braided monoidal category.

4 Duoidales and produoidal \mathcal{V} -categories

Recall the two following definitions and immediately following example from [10] where \mathcal{M} is a monoidal bicategory.

Definition 5 A *pseudomonoid* A in \mathcal{M} is an object A of \mathcal{M} together with multiplication and unit morphisms $\mu : A \otimes A \rightarrow A$, $\eta : I \rightarrow A$, and invertible 2-cells $a : \mu(\mu \otimes 1) \Rightarrow \mu(1 \otimes \mu)$, $\ell : \mu(\eta \otimes 1) \Rightarrow 1$, and $r : \mu(1 \otimes \eta) \Rightarrow 1$ satisfying the coherence conditions given in [10].

Definition 6 A (lax-)morphism f between pseudomonoids A and B in \mathcal{M} is a morphism $f : A \longrightarrow B$ equipped with

$$\begin{array}{ccc} M \otimes M & \xrightarrow{\mu} & M \\ f \otimes f \downarrow & \xRightarrow{\varphi} & \downarrow f \\ N \otimes N & \xrightarrow{\mu} & N \end{array}$$

and

$$\begin{array}{ccc} I & \xrightarrow{\eta} & M \\ & \xRightarrow{\varphi_0} & \downarrow f \\ & \xrightarrow{\eta} & N \end{array}$$

subject to three axioms.

Example If \mathcal{M} is the cartesian closed 2-category of categories, functors, and natural transformations then a monoidal category is precisely a pseudomonoid in \mathcal{M} .

This example motivates calling a pseudomonoid in a monoidal bicategory \mathcal{M} a *monoidale* (short for a monoidal object of \mathcal{M}). A morphism $f : M \rightarrow N$ of monoidales is then a morphism of pseudomonoids (i.e. a monoidal morphism between monoidal objects). We write $\text{Mon}(\mathcal{M})$ for the 2-category of monoidales in \mathcal{M} , monoidal morphisms, and monoidal 2-cells. If \mathcal{M} is symmetric monoidal then so is $\text{Mon}(\mathcal{M})$.

Definition 7 A *duoidale* F in \mathcal{M} is an object F together with two monoidale structures

$$* : F \otimes F \longrightarrow F, \quad J : I \longrightarrow F \quad (4.1)$$

$$\circ : F \otimes F \longrightarrow F, \quad \mathbf{1} : I \longrightarrow F \quad (4.2)$$

such that \circ and $\mathbf{1}$ are monoidal morphisms with respect to $*$ and J .

Remark If $\mathcal{M} = \mathcal{V}\text{-Cat}$ then a duoidale in \mathcal{M} is precisely a duoidal \mathcal{V} -category.

Let $\mathcal{M} = \mathcal{V}\text{-Mod}$ be the symmetric monoidal bicategory of \mathcal{V} -categories, \mathcal{V} -modules, and \mathcal{V} -module morphisms. By Proposition 8, there is a symmetric monoidal pseudofunctor

$$\mathcal{M}(\mathcal{I}, -) : \mathcal{M} \longrightarrow \mathcal{V}\text{-Cat}$$

defined by taking a \mathcal{V} -category \mathcal{A} to the \mathcal{V} -category $[\mathcal{A}^{\text{op}}, \mathcal{V}]$ of \mathcal{V} -functors and \mathcal{V} -natural transformations.

Definition 8 A *produoidal \mathcal{V} -category* is a duoidale in $\mathcal{V}\text{-Mod}$.

If \mathcal{F} is a produoidal \mathcal{V} -category then there are \mathcal{V} -modules

$$\begin{aligned} S : \mathcal{F} \otimes \mathcal{F} &\multimap \mathcal{F}, & H : \mathcal{I} &\multimap \mathcal{F}, \\ R : \mathcal{F} \otimes \mathcal{F} &\multimap \mathcal{F}, & K : \mathcal{I} &\multimap \mathcal{F}, \end{aligned}$$

where R and K are monoidal with respect to S so that there are 2-cells γ , δ , and τ :

$$\begin{array}{ccc} \mathcal{F} \otimes \mathcal{F} \otimes \mathcal{F} \otimes \mathcal{F} & \xrightarrow{1 \otimes c \otimes 1} & \mathcal{F} \otimes \mathcal{F} \otimes \mathcal{F} \otimes \mathcal{F} \xrightarrow{S \otimes S} \mathcal{F} \otimes \mathcal{F} \\ \downarrow R \otimes R & \searrow \gamma & \downarrow R \\ \mathcal{F} \otimes \mathcal{F} & \xrightarrow{S} & \mathcal{F} \end{array}$$

$$\begin{array}{ccc} \mathcal{I} & \simeq & \mathcal{I} \otimes \mathcal{I} \xrightarrow{H \otimes H} \mathcal{F} \otimes \mathcal{F} \\ & \searrow H & \downarrow R \\ & & \mathcal{F} \end{array} \quad \begin{array}{ccc} \mathcal{I} \otimes \mathcal{I} & \xrightarrow{\mu} & \mathcal{I} \\ \downarrow K \otimes K & & \downarrow K \\ \mathcal{F} \otimes \mathcal{F} & \xrightarrow{S} & \mathcal{F} \end{array}$$

$$\begin{array}{ccc} & H & \\ & \downarrow & \\ \mathcal{I} & \Downarrow \tau & \mathcal{F} \\ & \downarrow K & \end{array}$$

compatible with the two pseudomonoid structures. By composition of \mathcal{V} -modules these 2-cells have component morphisms

$$\begin{aligned} & \int^{X,Y} R(X; A, B) \otimes R(Y; C, D) \otimes S(E; X, Y) \\ & \quad \downarrow \gamma \\ & \int^{U,V} S(U; A, C) \otimes S(V; B, D) \otimes R(E; U, V) \\ \\ H(A) & \xrightarrow{\delta} \int^{X,Y} H(X) \otimes H(Y) \otimes R(A; X, Y) \\ \\ \int^{X,Y} K(X) \otimes K(Y) \otimes S(A; X, Y) & \xrightarrow{\mu} K(A) \\ \\ H(A) & \xrightarrow{\tau} K(A) \end{aligned}$$

in \mathcal{V} .

Given any duoidal \mathcal{V} -category \mathcal{F} we obtain a produoidal \mathcal{V} -category structure on \mathcal{F} by setting

$$S(A; B, C) = \mathcal{F}(A, B * C)$$

and

$$R(A; B, C) = \mathcal{F}(A, B \circ C)$$

that is, we pre-compose the \mathcal{V} -valued hom of \mathcal{F} with (3.1) and (3.2) of Definition 1.

Proposition 9 *If \mathcal{F} is a produoidal \mathcal{V} -category then $\mathcal{M}(\mathcal{I}, \mathcal{F}) = [\mathcal{F}^{\text{op}}, \mathcal{V}]$ is a duoidal \mathcal{V} -category.*

Proof Consider the \mathcal{V} -category of \mathcal{V} -functors and \mathcal{V} -natural transformations $[\mathcal{F}^{\text{op}}, \mathcal{V}]$. The two monoidal structures on \mathcal{F} translate to two monoidal structures on $[\mathcal{F}^{\text{op}}, \mathcal{V}]$ by Day-convolution

$$(M * N)(A) = \int^{X,Y} S(A; X, Y) \otimes M(X) \otimes N(Y) \quad (4.3)$$

$$(M \circ N)(B) = \int^{U,V} R(B; U, V) \otimes M(U) \otimes N(V) \quad (4.4)$$

such that the duoidal 2-cell structure morphisms lift to give a duoidal \mathcal{V} -category. More specifically the maps $(\gamma, \delta, \mu, \tau)$ lift to $[\mathcal{F}^{\text{op}}, \mathcal{V}]$ and satisfy the axioms (3.3), (3.4), (3.5) and (3.6) in Definition 1. Demonstrating the lifting and commutativity of the requisite axioms uses iterated applications of the \mathcal{V} -enriched Yoneda lemma and Fubini's interchange theorem as in [17]. ■

Our final theorem for this section permits us to apply the theory of categories enriched in a duoidal \mathcal{V} -category \mathcal{F} even if the monoidal structures on \mathcal{F} are not closed.

Theorem 10 *Let \mathcal{F} be a duoidal \mathcal{V} -category. The Yoneda embedding $y : \mathcal{F} \rightarrow [\mathcal{F}^{\text{op}}, \mathcal{V}]$ gives $[\mathcal{F}^{\text{op}}, \mathcal{V}]$ as the duoidal cocompletion of \mathcal{F} with both monoidal structures closed.*

Proof This theorem is essentially an extension of some results of Im and Kelly in [14] which themselves are largely extensions of results in [8] and [17]. In particular, if \mathcal{A} is a monoidal \mathcal{V} -category then $\hat{\mathcal{A}} = [\mathcal{A}^{\text{op}}, \mathcal{V}]$ is the free monoidal closed completion with the convolution monoidal structure. If \mathcal{F} is a duoidal \mathcal{V} -category then, by Proposition 4.1 of [14], the monoidal structures $*$ and \circ on \mathcal{F} give two monoidal biclosed structures on $\hat{\mathcal{F}} = [\mathcal{F}^{\text{op}}, \mathcal{V}]$ with the corresponding Yoneda embeddings strong monoidal functors. As per [14] the monoidal products are given by Day convolution

$$P \hat{*} Q = \int^{A,B} P(A) \otimes Q(B) \otimes \mathcal{F}(-, A * B) \quad (4.5)$$

$$P \hat{\circ} Q = \int^{A,B} P(A) \otimes Q(B) \otimes \mathcal{F}(-, A \circ B) \quad (4.6)$$

as the left Kan-extension of $y \otimes y$ along the composites $y*$ and $y\circ$ respectively. Write \hat{J} and $\hat{1}$ for the tensor units $y(J) = \mathcal{F}(-, J)$ and $y(\mathbf{1}) = \mathcal{F}(-, \mathbf{1})$ respectively. The duoidal data $(\gamma, \mu, \delta, \tau)$ lifts directly to give duoidal data $(\hat{\gamma}, \hat{\mu}, \hat{\delta}, \hat{\tau})$ for $\hat{\mathcal{F}}$. ■

5 Enrichment in a duoidal \mathcal{V} -category base

Let \mathcal{F} be a duoidal \mathcal{V} -category. There is a 2-category $\mathcal{F}_h\text{-Cat}$ of \mathcal{F}_h -categories, \mathcal{F}_h -functors, and \mathcal{F}_h -natural transformations in the usual Eilenberg-Kelly sense; see [17]. We write \mathcal{J} for the one-object \mathcal{F}_h -category whose hom is the horizontal unit J in \mathcal{F} .

Let \mathcal{A} and \mathcal{B} be \mathcal{F}_h -categories and define $\mathcal{A} \circ \mathcal{B}$ to be the \mathcal{F}_h -category with objects pairs (A, B) and hom-objects $(\mathcal{A} \circ \mathcal{B})((A, B), (A', B')) = \mathcal{A}(A, A') \circ$

$\mathcal{B}(B, B')$ in \mathcal{F}_h . Composition is defined using the middle of four map γ as follows

$$\begin{array}{c}
(\mathcal{A} \circ \mathcal{B})((A', B'), (A'', B'')) * (\mathcal{A} \circ \mathcal{B})((A, B), (A', B')) \\
\cong \downarrow \\
(\mathcal{A}(A', A'') \circ \mathcal{B}(B', B'')) * (\mathcal{A}(A, A') \circ \mathcal{B}(B, B')) \\
\gamma \downarrow \\
(\mathcal{A}(A', A'') * \mathcal{A}(A, A')) \circ (\mathcal{B}(B', B'') * \mathcal{B}(B, B')) \\
\text{comp} \circ \text{comp} \downarrow \\
\mathcal{A}(A, A'') \circ \mathcal{B}(B, B'') \\
\cong \downarrow \\
(\mathcal{A} \circ \mathcal{B})((A, B), (A'', B'')).
\end{array}$$

Identities are given by the composition

$$J \xrightarrow{\delta} J \circ J \xrightarrow{\hat{id}_A \circ \hat{id}_B} \mathcal{A}(A, A) \circ \mathcal{B}(B, B).$$

The monoidal unit is the \mathcal{F}_h -category $\bar{\mathbf{1}}$ consisting of a single object \bullet and hom-object $\bar{\mathbf{1}}(\bullet, \bullet) = \mathbf{1}$.

Checking the required coherence conditions proves the following result of [2].

Proposition 11 *The \circ monoidal structure on \mathcal{F}_h lifts to a monoidal structure on the 2-category $\mathcal{F}_h\text{-Cat}$.*

We write $\mathcal{F}\text{-Cat}$ for the monoidal 2-category $\mathcal{F}_h\text{-Cat}$ with \circ as the tensor product.

Let \mathcal{F} be a duoidal \mathcal{V} -category such that the horizontal monoidal structure $*$ is left-closed. That is, we have

$$\mathcal{F}(X * Y, Z) \cong \mathcal{F}(X, [Y, Z])$$

with the “evaluation” counit $ev : [Y, Z] * Y \rightarrow Z$.

This gives \mathcal{F}_h as an \mathcal{F}_h -category in the usual way by defining the composition operation $[Y, Z] * [X, Y] \rightarrow [X, Z]$ as corresponding to

$$([Y, Z] * [X, Y]) * X \cong [Y, Z] * ([X, Y] * X) \xrightarrow{1 * ev} [Y, Z] * Y \xrightarrow{ev} Z$$

and identities $\hat{id}_X : J \rightarrow [X, X]$ as corresponding to $\ell : J * X \rightarrow X$.

The duoidal structure of \mathcal{F} provides a way of defining $[X, X'] \circ [Y, Y'] \rightarrow [X \circ Y, X' \circ Y']$ using the the middle-of-four interchange map:

$$\begin{array}{ccc}
([X, X'] \circ [Y, Y']) * (X \circ Y) & \xrightarrow{\quad} & X' \circ Y' \\
\gamma \downarrow & \nearrow ev \circ ev & \\
([X, X'] * X) \circ ([Y, Y'] * Y) & &
\end{array} \tag{5.1}$$

The above shows that \mathcal{F} is a monoidale (pseudo-monoid) in the category of \mathcal{F}_h -categories with multiplication given by the \mathcal{F}_h -functor $\hat{\circ} : \mathcal{F}_h \circ \mathcal{F}_h \longrightarrow \mathcal{F}_h$ as defined.

Let $\text{Mon}(\mathcal{F}_h)$ be the category of (horizontal) monoids $(M, \mu : M * M \longrightarrow M, \eta : J \longrightarrow M)$ in \mathcal{F}_h . Let M and N be objects of $\text{Mon}(\mathcal{F}_h)$ and define the monoid multiplication map of $M \circ N$ to be the composition

$$(M \circ N) * (M \circ N) \xrightarrow{\gamma} (M * M) \circ (N * N) \xrightarrow{\mu \circ \mu} M \circ N$$

and the unit to be

$$J \xrightarrow{\delta} J \circ J \xrightarrow{\eta \circ \eta} M \circ N .$$

This tensor product of monoids is the restriction to one-object \mathcal{F}_h -categories of the tensor of \mathcal{F} -Cat. So we have the following result which was also observed in [1].

Proposition 12 *The monoidal structure \circ on \mathcal{F} lifts to a monoidal structure on the category $\text{Mon}(\mathcal{F}_h)$.*

We write $\text{Mon } \mathcal{F}$ for the monoidal category $\text{Mon}(\mathcal{F}_h)$ with \circ .

Remark A monoid in $(\text{Mon } \mathcal{F})^{\text{op}}$ is precisely a bimonoid in \mathcal{F} .

6 The Tannaka adjunction revisited

Let \mathcal{F} be a horizontally left closed duoidal \mathcal{V} -category. Each object M of \mathcal{F} determines an \mathcal{F}_h -functor

$$- * M : \mathcal{F}_h \longrightarrow \mathcal{F}_h$$

defined on objects by $A \mapsto A * M$ and on homs by taking

$$- * M : [A, B] \longrightarrow [A * M, B * M] \quad (6.1)$$

to correspond to

$$[A, B] * (A * M) \cong ([A, B] * A) * M \xrightarrow{ev * 1} B * M .$$

If M is a monoid in \mathcal{F}_h then $- * M$ becomes a monad in $\mathcal{F}_h\text{-Cat}$ in the usual way.

We write \mathcal{F}^{*M} for the Eilenberg-Moore \mathcal{F}_h -category of algebras for the \mathcal{F}_h -monad $- * M$; see [18] and [22]. It is the \mathcal{F}_h -category of right M -modules in \mathcal{F} . If \mathcal{F} has equalizers then \mathcal{F}^{*M} is assured to exist; the \mathcal{F}_h -valued hom is the equalizer of the pair

$$\begin{array}{ccc} [A, B] & \xrightarrow{[\alpha, 1]} & [A * M, B] \\ & \searrow - * M & \nearrow [1, \beta] \\ & [A * M, B * M] & \end{array} \quad (6.2)$$

where $\alpha : A * M \longrightarrow A$ and $\beta : B * M \longrightarrow B$ are the actions of A and B as objects of \mathcal{F}^{*M} .

Let $U_M : \mathcal{F}^{*M} \longrightarrow \mathcal{F}_h$ denote the underlying \mathcal{F}_h -functor which forgets the action and whose effect on homs is the equalizer of (6.2). There is an \mathcal{F}_h -natural transformation

$$\chi : U_M * M \longrightarrow U_M \quad (6.3)$$

which is the universal action of the monad $- * M$; its component at A in \mathcal{F}^{*M} is precisely the action $\alpha : A * M \longrightarrow A$ of A .

An aspect of the strong enriched Yoneda Lemma is the \mathcal{F}_h -natural isomorphism

$$\mathcal{F}^{*M}(M, B) \cong U_M B. \quad (6.4)$$

In this special case, the result comes from the equalizer

$$B \xrightarrow{\hat{\beta}} [M, B] \xrightleftharpoons[[1, \beta](-*M)]{[\mu, 1]} [M * M, B].$$

In other words, the \mathcal{F}_h -functor U_M is representable with M as the representing object.

Each \mathcal{F}_h -functor $U : \mathcal{A} \longrightarrow \mathcal{F}_h$ defines a functor

$$U * - : \mathcal{F} \longrightarrow \mathcal{F}_h\text{-Cat}(\mathcal{A}, \mathcal{F}_h) \quad (6.5)$$

taking $X \in \mathcal{F}$ to the composite \mathcal{F}_h -functor

$$\mathcal{A} \xrightarrow{U} \mathcal{F}_h \xrightarrow{-*X} \mathcal{F}_h$$

and $f : X \longrightarrow Y$ to the \mathcal{F}_h -natural transformation $U * f$ with components

$$1 * f : U A * X \longrightarrow U A * Y.$$

We shall call $U : \mathcal{A} \longrightarrow \mathcal{F}_h$ *tractable* when the functor $U * -$ has a right adjoint denoted

$$\{U, -\} : \mathcal{F}_h\text{-Cat}(\mathcal{A}, \mathcal{F}_h) \longrightarrow \mathcal{F}. \quad (6.6)$$

This means that morphisms $t : X \longrightarrow \{U, V\}$ are in natural bijection with \mathcal{F}_h -natural transformations $\theta : U * X \longrightarrow V$.

Let us examine what \mathcal{F}_h -naturality of $\theta : U * X \longrightarrow V$ means. By definition it means commutativity of

$$\begin{array}{ccc} & [VA, VB] & \\ V_{A,B} \nearrow & & \searrow [\theta_A, 1] \\ \mathcal{A}(A, B) & & [UA * X, VB] \\ U_{A,B} \downarrow & & \uparrow [1, \theta_B] \\ [UA, UB] & \xrightarrow{-*X} & [UA * X, UB * X]. \end{array} \quad (6.7)$$

This is equivalent to the module-morphism condition

$$\begin{array}{ccc} \mathcal{A}(A, B) * UA * X & \xrightarrow{1 * \theta_A} & \mathcal{A}(A, B) * VA \\ \overline{U}_{A, B} \downarrow & & \downarrow \nabla_{A, B} \\ UB * X & \xrightarrow{\theta_B} & VB \end{array} \quad (6.8)$$

under left closedness of \mathcal{F}_h . Notice that tractability of an object Z of \mathcal{F} , regarded as an \mathcal{F}_h -functor $\lceil Z \rceil : \mathcal{J} \rightarrow \mathcal{F}_h$, is equivalent to the existence of a horizontal right hom $\{Z, -\}$:

$$\mathcal{F}(X, \{Z, Y\}) \cong \mathcal{F}(Z * X, Y). \quad (6.9)$$

Assuming all of the objects UA and $\mathcal{A}(A, B)$ in \mathcal{F} are tractable, we can rewrite (6.8) in the equivalent form

$$\begin{array}{ccccc} & \{UA, VA\} & \xrightarrow{\{1, \hat{V}_{AB}\}} & \{UA, \{\mathcal{A}(A, B), VB\}\} & \\ \nearrow \hat{\theta}_A & & & \downarrow \cong & \\ X & & & & \\ \searrow \hat{\theta}_B & \{UB, VB\} & \xrightarrow{\{\hat{U}_{AB}, 1\}} & \{\mathcal{A}(A, B) * UA, VB\} & \end{array} \quad (6.10)$$

Proposition 13 *If \mathcal{F} is a complete, horizontally left and right closed, duoidal \mathcal{V} -category and \mathcal{A} is a small \mathcal{F}_h -category then every \mathcal{F}_h -functor $U : \mathcal{A} \rightarrow \mathcal{F}_h$ is tractable.*

However, some U can still be tractable even when \mathcal{A} is not small.

Proposition 14 (*Yoneda Lemma*) *If $U : \mathcal{A} \rightarrow \mathcal{F}_h$ is an \mathcal{F}_h -functor represented by an object K of \mathcal{A} then U is tractable and*

$$\{U, V\} \cong VK.$$

Proof By the “weak Yoneda Lemma” (see [17]) we have

$$\mathcal{F}_h\text{-Cat}(U * X, V) \cong \mathcal{F}_h\text{-Cat}(U, [X, V]) \cong \mathcal{F}(J, [X, VK]) \cong \mathcal{F}(X, VK). \quad \blacksquare$$

Consider the 2-category $\mathcal{F}_h\text{-Cat} \downarrow^{\text{ps}} \mathcal{F}_h$ defined as follows. The objects are \mathcal{F}_h -functors $U : \mathcal{A} \rightarrow \mathcal{F}_h$. The morphisms $(T, \tau) : U \rightarrow V$ are triangles

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{T} & \mathcal{B} \\ & \tau \cong & \\ U \searrow & & \swarrow V \\ & \mathcal{F}_h & \end{array} \quad (6.11)$$

in $\mathcal{F}_h\text{-Cat}$. The 2-cells $\theta : (T, \tau) \Rightarrow (S, \sigma)$ are \mathcal{F}_h -natural transformations $\theta : T \Rightarrow S$ such that

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{T} & \mathcal{B} \\ \downarrow \theta & & \downarrow \tau \\ \mathcal{A} & \xrightarrow{S} & \mathcal{B} \\ \downarrow U & \xrightarrow{\cong} & \downarrow V \\ \mathcal{F}_h & & \mathcal{F}_h \end{array} = \begin{array}{ccc} \mathcal{A} & \xrightarrow{T} & \mathcal{B} \\ & \tau \cong & \\ \downarrow U & & \downarrow V \\ \mathcal{F}_h & & \mathcal{F}_h \end{array} . \quad (6.12)$$

We define a *vertical tensor product* \circ on the 2-category $\mathcal{F}_h\text{-Cat} \downarrow^{\text{ps}} \mathcal{F}_h$ making it a monoidal 2-category, which we denote by $\mathcal{F}\text{-Cat} \downarrow^{\text{ps}} \mathcal{F}$. For \mathcal{F}_h -functors $U : \mathcal{A} \rightarrow \mathcal{F}_h$ and $V : \mathcal{B} \rightarrow \mathcal{F}_h$, define $U \circ V : \mathcal{A} \circ \mathcal{B} \rightarrow \mathcal{F}_h$ to be the composite

$$\mathcal{A} \circ \mathcal{B} \xrightarrow{U \circ V} \mathcal{F}_h \circ \mathcal{F}_h \xrightarrow{\hat{\circ}} \mathcal{F}_h . \quad (6.13)$$

The unit object is $\lceil 1 \rceil : \bar{\mathbf{1}} \rightarrow \mathcal{F}_h$. The associativity constraints are explained by the diagram

$$\begin{array}{ccc} (\mathcal{A} \circ \mathcal{B}) \circ \mathcal{C} & \xrightarrow{\cong} & \mathcal{A} \circ (\mathcal{B} \circ \mathcal{C}) \\ \downarrow (U \circ V) \circ W & & \downarrow U \circ (V \circ W) \\ (\mathcal{F}_h \circ \mathcal{F}_h) \circ \mathcal{F}_h & \xrightarrow{\cong} & \mathcal{F}_h \circ (\mathcal{F}_h \circ \mathcal{F}_h) \\ \downarrow \hat{\circ} \circ 1 & \xrightarrow{a} & \downarrow 1 \circ \hat{\circ} \\ \mathcal{F}_h \circ \mathcal{F}_h & & \mathcal{F}_h \circ \mathcal{F}_h \\ & \searrow \hat{\circ} & \swarrow \hat{\circ} \\ & \mathcal{F}_h & \end{array} \quad (6.14)$$

where a is the associativity constraint for the vertical structure on \mathcal{F} . The unit constraints are similar.

Remark We would like to emphasise that, although there are conceivable 2-cells for $\text{Mon } \mathcal{F}$ as a sub-2-category of $\mathcal{F}_h\text{-Cat}$ (see [22]), we are only regarding $\text{Mon } \mathcal{F}$ as a monoidal category, not a monoidal 2-category.

Next we specify a monoidal functor

$$\text{mod} : (\text{Mon } \mathcal{F})^{\text{op}} \longrightarrow \mathcal{F}\text{-Cat} \downarrow^{\text{ps}} \mathcal{F} . \quad (6.15)$$

For each monoid M in \mathcal{F}_h , we put

$$\text{mod } M = (U_M : \mathcal{F}^{*M} \longrightarrow \mathcal{F}_h) .$$

For a monoid morphism $f : N \rightarrow M$, we define

$$\begin{array}{ccc} \mathcal{F}^{*M} & \xrightarrow{\text{mod } f} & \mathcal{F}^{*N} \\ & \searrow U_M & \swarrow U_N \\ & \mathcal{F} & \end{array} = \quad (6.16)$$

by

$$(\text{mod } f)(A * M \xrightarrow{\alpha} A) = (A * N \xrightarrow{1*f} A * M \xrightarrow{\alpha} A).$$

To see that $\text{mod } f$ is an \mathcal{F}_h -functor, we recall the equalizer of (6.2) and point to the following diagram in which the empty regions commute.

$$\begin{array}{ccccc}
& & [A * M, B] & & \\
& \nearrow [\alpha, 1] & \uparrow [1, \beta] & \searrow [1*f, 1] & \\
[A, B] & \xrightarrow{\quad - * M \quad} & [A * M, B * M] & \xrightarrow{\quad [1*f, 1] \quad} & [A * N, B] \\
& \searrow - * N & & \nearrow [1, \beta] & \\
& & [A * N, B * N] & \xrightarrow{\quad [1, 1*f] \quad} & [A * N, B * M]
\end{array}$$

(6.2)

Alternatively, we could use the universal property of $\text{mod } N$ as the universal action of the monad $- * N$ on \mathcal{F} .

For the monoidal structure on mod , we define an \mathcal{F}_h -functor $\Phi_{M,N}$ making the square

$$\begin{array}{ccc}
\mathcal{F}^* M \circ \mathcal{F}^* N & \xrightarrow{\Phi_{M,N}} & \mathcal{F}^{*(M \circ N)} \\
U_M \circ U_N \downarrow & & \downarrow U_{M \circ N} \\
\mathcal{F}_h \circ \mathcal{F}_h & \xrightarrow{\quad \circ \quad} & \mathcal{F}_h
\end{array} \tag{6.17}$$

commute; put

$$\begin{aligned}
\Phi_{M,N}(A * M \xrightarrow{\alpha} A, B * N \xrightarrow{f} B) = \\
((A \circ B) * (M \circ N) \xrightarrow{\gamma} (A * M) \circ (B * N) \xrightarrow{\alpha \circ \beta} A \circ B)
\end{aligned}$$

and use the universal property of $\text{mod}(M \circ N)$ to define $\Phi_{M,N}$ on homs.

For tractable $U : \mathcal{A} \rightarrow \mathcal{F}_h$, we have an evaluation \mathcal{F}_h -natural transformation

$$ev : U * \{U, V\} \longrightarrow V,$$

corresponding under the adjunction (6.6), to the identity of $\{U, V\}$. We have a “composition morphism”

$$\mu : \{U, V\} * \{V, W\} \longrightarrow \{U, W\}$$

corresponding to the composite

$$U * \{U, V\} * \{V, W\} \xrightarrow{ev * 1} V * \{V, W\} \xrightarrow{ev} W.$$

In particular,

$$\mu : \{U, U\} * \{U, U\} \longrightarrow \{U, U\}$$

together with

$$\eta : J \longrightarrow \{U, U\},$$

corresponding to $U * J \cong U$, gives $\{U, U\}$ the structure of a monoid, denoted U , in \mathcal{F}_h .

Proposition 15 *For each tractable \mathcal{F}_h -functor $U : \mathcal{A} \rightarrow \mathcal{F}_h$, there is an equivalence of categories*

$$(\text{Mon } \mathcal{F}_h)(M, \text{end } U) \simeq (\mathcal{F}_h\text{-Cat } \downarrow^{\text{ps}} \mathcal{F}_h)(U, \text{mod } M)$$

pseudonatural in monoids M in \mathcal{F}_h .

Proof Morphisms $t : M \rightarrow \text{end } U$ in \mathcal{F} are in natural bijection (using (6.6)) with \mathcal{F}_h -natural transformations $\theta : U * M \rightarrow U$. It is easy to see that t is a monoid morphism if and only if θ is an action of the monad $- * M$ on $U : \mathcal{A} \rightarrow \mathcal{F}_h$. By the universal property of the Eilenberg-Moore construction [22], such actions are in natural bijection with liftings of U to \mathcal{F}_h -functors $\mathcal{A} \rightarrow \mathcal{F}^{*M}$. This describes a bijection between $(\text{Mon } \mathcal{F}_h)(M, \text{end } U)$ and the full subcategory of $(\mathcal{F}_h\text{-Cat } \downarrow^{\text{ps}} \mathcal{F}_h)(U, \text{mod } M)$ consisting of the morphisms

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{T} & \mathcal{F}^{*M} \\ & \searrow U & \swarrow U_M \\ & \mathcal{F}_h & \end{array} \quad \begin{array}{c} \tau \\ \cong \end{array}$$

for which τ is an identity. It remains to show that every general such morphism (T, τ) is isomorphic to one for which τ is an identity. However, each (T, τ) determines an action

$$U * M \xrightarrow{\tau} U_M T * M = (U_M * M) T \xrightarrow{\chi^T} U_M T \xrightarrow{\tau^{-1}} U$$

of the monad $- * M$ on U . By the universal property, we induce a morphism

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{T'} & \mathcal{F}^{*M} \\ & \searrow U & \swarrow U_M \\ & \mathcal{F}_h & \end{array} \quad \begin{array}{c} = \end{array}$$

and an invertible 2-cell $(T, \tau) \cong (T', 1)$ in $\mathcal{F}_h\text{-Cat } \downarrow^{\text{ps}} \mathcal{F}_h$. ■

In other words, we have a biadjunction

$$(\text{Mon } \mathcal{F}_h)^{\text{op}} \begin{array}{c} \xleftarrow{\text{end}} \\ \perp \\ \xrightarrow{\text{mod}} \end{array} \mathcal{F}_h\text{-Cat } \downarrow_{\text{tract}}^{\text{ps}} \mathcal{F}_h \quad (6.18)$$

where the 2-category on the right has objects restricted to the tractable U . As a consequence, notice that end takes each 2-cell to an identity (since all 2-cells in $\text{Mod } \mathcal{F}_h$ are identities). Notice too from the notation that we are ignoring the monoidal structure in (6.18). This is because tractable U are not generally closed under the monoidal structure of $\mathcal{F}\text{-Cat } \downarrow^{\text{ps}} \mathcal{F}$.

Proposition 16 *Representable objects of $\mathcal{F}\text{-Cat } \downarrow^{\text{ps}} \mathcal{F}$ are closed under the monoidal structure.*

Proof

$$\begin{array}{ccc} \mathcal{A} \circ \mathcal{B} & \xrightarrow{\mathcal{A}(A, -) \circ \mathcal{B}(B, -)} & \mathcal{F}_h \circ \mathcal{F}_h \xrightarrow{\hat{\circ}} \mathcal{F}_h \\ & \searrow \cong & \nearrow \\ & (\mathcal{A} \circ \mathcal{B})((A, B), -) & \end{array}$$

and

$$\lceil \mathbf{1} \rceil = \bar{\mathbf{I}}(\bullet, -) : \bar{\mathbf{I}} \longrightarrow \mathcal{F}_h. \quad \blacksquare$$

Let $\mathcal{F}\text{-Cat} \downarrow_{\text{rep}}^{\text{ps}} \mathcal{F}$ denote the monoidal full sub-2-category of $\mathcal{F}\text{-Cat} \downarrow^{\text{ps}} \mathcal{F}$ consisting of the representable objects. The biadjunction (6.18) restricts to a biadjunction

$$\begin{array}{ccc} (\text{Mon } \mathcal{F}_h)^{\text{op}} & \xleftarrow[\text{mod}]{\text{end}} & \mathcal{F}_h\text{-Cat} \downarrow_{\text{rep}}^{\text{ps}} \mathcal{F}_h \\ & \perp & \end{array} \quad (6.19)$$

and we have already pointed out that mod is monoidal; see (6.17). In fact, we shall soon see that this is a monoidal biadjunction.

First note that, if $U : \mathcal{A} \longrightarrow \mathcal{F}_h$ is represented by K then we have a monoidal isomorphism

$$\text{end } U = \{U, U\} \stackrel{(14)}{\cong} UK \cong \mathcal{A}(K, K). \quad (6.20)$$

In particular, for a monoid M in \mathcal{F}_h , using Proposition 4, we obtain a monoid isomorphism

$$\text{end mod } M \cong M \quad (6.21)$$

which is in fact the counit for (6.19), confirming that mod is an equivalence on homs.

Proposition 17 *The 2-functor end in (6.19) is strong monoidal.*

Proof The isomorphism (6.20) gives

$$\begin{aligned} \text{end } (\mathcal{A} \circ \mathcal{B})((A, B), -) &\cong (\mathcal{A} \circ \mathcal{B})((A, B), (A, B)) \\ &\cong \mathcal{A}(A, A) \circ \mathcal{B}(B, B) \\ &\cong \text{end } \mathcal{A}(A, -) \circ \text{end } \mathcal{B}(B, -) \end{aligned}$$

and

$$\text{end } \bar{\mathbf{I}}(\bullet, -) \cong \bar{\mathbf{I}}(\bullet, \bullet) \cong \mathbf{1}. \quad \blacksquare$$

As previously remarked, a monoid in $(\text{Mon } \mathcal{F}_h)^{\text{op}}$ is precisely a bimonoid in \mathcal{F} ; see Definition 4. Since $\text{Mon } \mathcal{F}$ has discrete homs, these monoids are the same as pseudomonoids. The biadjunction (6.18) determines a biadjunction

$$\begin{array}{ccc} (\text{Bimon } \mathcal{F}_h)^{\text{op}} & \xleftarrow[\text{mod}]{\text{end}} & \text{Mon}_{\text{ps}}(\mathcal{F}\text{-Cat} \downarrow_{\text{rep}}^{\text{ps}} \mathcal{F}). \end{array} \quad (6.22)$$

A pseudomonoid in $\mathcal{F}\text{-Cat} \downarrow \mathcal{F}$ is a monoidal \mathcal{F}_h -category \mathcal{A} together with a strong monoidal \mathcal{F}_h -functor $U : \mathcal{A} \longrightarrow \mathcal{F}_h$ (where \mathcal{F}_h has $\hat{\circ}$ as the monoidal structure).

This leads to the following lifting to the duoidal setting of a result attributed to Bodo Pareigis (see [3], [4] and [5]).

Theorem 18 *For a horizontal monoid M in a duoidal \mathcal{V} -category \mathcal{F} , bimonoid structures on M are in bijection with isomorphism classes of monoidal structures on \mathcal{F}^{*M} such that $U_M : \mathcal{F}^{*M} \rightarrow \mathcal{F}$ is strong monoidal into the vertical structure on \mathcal{F} .*

Proof For any horizontal monoid M in \mathcal{F} we (in the order they appear) have (6.18), Proposition 17 and (6.21) giving

$$\begin{aligned} & (\mathcal{F}\text{-Cat} \downarrow^{\text{ps}} \mathcal{F})(\text{mod } M \circ \text{mod } M, \text{mod } M) \\ \cong & (\text{Mon } \mathcal{F})(M, \text{end } (\text{mod } M \circ \text{mod } M)) \\ \cong & (\text{Mon } \mathcal{F})(M, \text{end mod } M \circ \text{end mod } M) \\ \cong & (\text{Mon } \mathcal{F})(M, M \circ M) \end{aligned}$$

and

$$(\mathcal{F}\text{-Cat} \downarrow^{\text{ps}} \mathcal{F})(\ulcorner \mathbf{1} \urcorner, \text{mod } M) \simeq (\text{Mon } \mathcal{F})(M, \mathbf{1}).$$

By Proposition 17, each bimonoid structure on M yields a pseudomonoid structure on $\text{mod } M$; and each pseudomonoid structure on $\text{mod } M$ yields a bimonoid structure on $\text{end mod } M \cong M$. The above equivalences give the bijection of the Theorem. \blacksquare

7 Hopf bimonoids

We have seen that a bimonoid M in a duoidal \mathcal{V} -category \mathcal{F} leads to a monoidal \mathcal{F}_h -category \mathcal{F}^{*M} of right M -modules. In this section, we are interested in when \mathcal{F}^{*M} is closed. We lean heavily on papers [6] and [7].

A few preliminaries from [21] adapted to \mathcal{F}_h -categories are required. For an \mathcal{F}_h -category \mathcal{A} , a *right \mathcal{A} -module* $W : \mathcal{J} \multimap \mathcal{A}$ is a family of objects WA of \mathcal{F} indexed by the objects A of \mathcal{A} and a family

$$W_{AB} : WA * \mathcal{A}(B, A) \multimap WB$$

of morphisms of \mathcal{F} indexed by pairs of objects A, B of \mathcal{A} , satisfying the action conditions. For modules $W, W' : \mathcal{J} \multimap \mathcal{A}$, define $[W, W']$ to be the limit as below when it exists in \mathcal{F} .

$$\begin{array}{ccc} [WA, W'A] & \xrightarrow{- * \mathcal{A}(B, A)} & [WA * \mathcal{A}(B, A), W'A * \mathcal{A}(B, A)] \\ \swarrow \text{---} & & \downarrow [1, W'_{AB}] \\ [W, W'] & & \\ \searrow \text{---} & & \downarrow \\ [WB, W'B] & \xrightarrow{[W_{AB}, 1]} & [WA * \mathcal{A}(B, A), W'B] \end{array} \quad (7.1)$$

Example A monoid M in \mathcal{F}_h can be regarded as a one object \mathcal{F}_h -category. A right M -module $A : \mathcal{J} \multimap M$ is precisely an object of \mathcal{F}^{*M} .

Example For any \mathcal{F}_h -functor $S : \mathcal{A} \rightarrow \mathcal{X}$ and object X of \mathcal{X} , we obtain a right \mathcal{A} -module $\mathcal{X}(S, X) : \mathcal{J} \multimap \mathcal{A}$ defined by the objects $\mathcal{X}(SA, X)$ of \mathcal{F} and the morphisms

$$\mathcal{X}(SA, X) * \mathcal{A}(B, A) \xrightarrow{1 * S_{BA}} \mathcal{X}(SA, X) * \mathcal{X}(SB, SA) \xrightarrow{\text{comp}} \mathcal{X}(SB, X).$$

Recall from [21] that the *colimit* $\text{colim}(W, S)$ of $S : \mathcal{A} \rightarrow \mathcal{X}$ weighted by $W : \mathcal{J} \rightarrow \mathcal{A}$ is an object of \mathcal{X} for which there is an \mathcal{F}_h -natural isomorphism

$$\mathcal{X}(\text{colim}(W, S), X) \cong [W, \mathcal{X}(S, X)] \quad (7.2)$$

By Yoneda, such an isomorphism is induced by the module morphism

$$\lambda : W \rightarrow \mathcal{X}(S, \text{colim}(W, S)) . \quad (7.3)$$

The \mathcal{F}_h -functor $S : \mathcal{A} \rightarrow \mathcal{X}$ is *dense* when $\lambda = 1 : \mathcal{X}(S, Y) \rightarrow \mathcal{X}(S, Y)$ induces

$$\text{colim}(\mathcal{X}(S, Y), S) \cong Y \quad (7.4)$$

for all Y in \mathcal{X} .

Proposition 19 *The \mathcal{F}_h -functor $\ulcorner J \urcorner : \mathcal{J} \rightarrow \mathcal{F}_h$ is dense.*

Proof From (7.2) we see that

$$[Y, X] \cong [J, [Y, X]]$$

implies

$$\text{colim}([J, Y], J) \cong Y,$$

which is (7.4) in this case. \blacksquare

Another element of our analysis is to recast the middle-of-four interchange morphisms as a 2-cell in $\mathcal{F}_h\text{-Cat}$.

Proposition 20 *The family of morphisms*

$$\gamma : (X \circ Y) * (C \circ D) \longrightarrow (X * C) \circ (Y * D)$$

defines an \mathcal{F}_h -natural transformation

$$\begin{array}{ccc} & \mathcal{F} & \\ \hat{\circ} \nearrow & & \searrow \circ * (C \circ D) \\ \mathcal{F} \circ \mathcal{F} & \Downarrow \gamma & \mathcal{F} \\ \circ * (C) \circ \circ * (D) \searrow & & \nearrow \hat{\circ} \\ & \mathcal{F} \circ \mathcal{F} & \end{array}$$

for all objects C and D of \mathcal{F} .

Proof Regard the commutative diagram

$$\begin{array}{ccc} ([X, U] \circ [Y, V]) * ((X * C) \circ (Y * D)) & \xleftarrow{1 * \gamma} & ([X, U] \circ [Y, V]) * (X \circ Y) * (C \circ D) \\ \gamma \downarrow & (3.3) & \downarrow \gamma * 1 \\ ([X, U] * X * C) \circ ([Y, V] * Y * D) & \xleftarrow{\gamma} & (([U, X] * X) \circ ([Y, V] * Y)) * (C \circ D) \\ (ev * C) \circ (ev * D) \downarrow & \text{naturality} & \downarrow (ev \circ ev) * 1 \\ (U * C) \circ (V * D) & \xleftarrow{\gamma} & (U \circ V) * (C \circ D) \end{array}$$

in which we have written as if $*$ were strict. \blacksquare

Proposition 21 Suppose $\theta : F \Rightarrow G : \mathcal{X} \rightarrow \mathcal{Y}$ is an \mathcal{F}_h -natural transformation between \mathcal{F}_h -functors F and G which preserve colimits weighted by $W : \mathcal{J} \multimap \mathcal{A}$. If each $\theta_{SA} : FSA \rightarrow GSA$ is invertible then so is

$$\theta_{\text{colim}(W,S)} : F \text{ colim}(W, S) \multimap G \text{ colim}(W, S) .$$

Proof

$$\begin{array}{ccc} F \text{ colim}(W, S) & \xrightarrow{\theta_{\text{colim}(W,S)}} & G \text{ colim}(W, S) \\ \cong \downarrow & & \downarrow \cong \\ \text{colim}(W, FS) & \xrightarrow{\theta_{\text{colim}(1, \theta_S)}} & \text{colim}(W, GS) . \end{array} \quad \blacksquare$$

Definition 9 For a bimonoid M in a duoidal category \mathcal{F} , the composite v_ℓ :

$$(J \circ M) * M \xrightarrow{1 * \delta} (J \circ M) * (M \circ M) \xrightarrow{\gamma} (J * M) \circ (M * M) \xrightarrow{\ell \circ \mu} M \circ M$$

is called the *left fusion morphism*. The composite v_r :

$$(M \circ J) * M \xrightarrow{1 * \delta} (M \circ J) * (M \circ M) \xrightarrow{\gamma} (M * M) \circ (J * M) \xrightarrow{\mu \circ \ell} M \circ M$$

is called the *right fusion morphism*. We call M *left Hopf* when v_ℓ is invertible and *right Hopf* when v_r is invertible. We call M *Hopf* when both v_ℓ and v_r are invertible.

Suppose \mathcal{A} and \mathcal{X} are monoidal \mathcal{F}_h -categories and $U : \mathcal{A} \rightarrow \mathcal{X}$ is a monoidal \mathcal{F}_h -functor. Writing \circ for the tensor and $\mathbf{1}$ for the tensor unit, we must have morphisms

$$\varphi : UA \circ UB \multimap U(A \circ B) \quad \text{and} \quad \varphi_0 : \mathbf{1} \multimap U\mathbf{1}$$

satisfying the usual Eilenberg-Kelly [12] conditions. Suppose \mathcal{A} and \mathcal{X} are left closed and write $\ell om(A, B)$ and $\ell om(X, Y)$ for the left homs. As pointed out by Eilenberg-Kelly, the monoidal structure φ, φ_0 is in bijection with *left closed structure*

$$\varphi^\ell : U \ell om(A, B) \circ UB \multimap U \ell om(UA, UB) \quad \text{and} \quad \varphi_0 : \mathbf{1} \multimap U\mathbf{1},$$

where φ^ℓ corresponds under the adjunction to the composite

$$U \text{ hom}(A, B) \circ UA \xrightarrow{\varphi} U(\text{hom}(A, B) \circ A) \xrightarrow{U_{ev}} UB$$

Following [11], we say U is *strong left closed* when both φ^ℓ and φ_0 are invertible.

Recall from [6] (and [7] for the enriched situation) that the Eilenberg-Moore (enriched) category for an opmonoidal monad T on \mathcal{X} is left closed and the forgetful $U_T : \mathcal{X}^T \rightarrow \mathcal{X}$ is strong left closed if and only if T is “left Hopf”. The monad T is *left Hopf* when the *left fusion morphism*

$$v_\ell(X, Y) : T(X \circ TY) \xrightarrow{\varphi} TX \circ T^2Y \xrightarrow{1 \circ \mu} TX \circ TY \quad (7.5)$$

is invertible for all X and Y . It is *right Hopf* when the *right fusion morphism*

$$v_r(X, Y) : T(TX \circ Y) \xrightarrow{\varphi} T^2X \circ TY \xrightarrow{\mu \circ 1} TX \circ TY \quad (7.6)$$

is invertible.

In particular, for a bimonoid M in \mathcal{F} , taking $T = - * M$, we see that $v_\ell(X, Y)$ is the composite

$$\begin{array}{ccc} (X \circ (Y * M)) * M & \xrightarrow{1 * \delta} & (X \circ (Y * M)) * (M \circ M) \\ \downarrow v_\ell(X, Y) & & \downarrow \gamma \\ & & (X * M) \circ ((Y * M) * M) \\ & & \cong \downarrow 1 \circ a \\ (X * M) \circ (Y * M) & \xleftarrow{1 \circ (1 * \mu)} & (X * M) \circ (Y * (M * M)) \end{array} \quad (7.7)$$

and that $v_r(X, Y)$ is

$$\begin{array}{ccc} ((X * M) \circ Y) * M & \xrightarrow{1 * \delta} & ((X * M) \circ Y) * (M \circ M) \\ \downarrow v_r(X, Y) & & \downarrow \gamma \\ & & ((X * M) * M) \circ (Y * M) \\ & & \cong \downarrow a \circ 1 \\ (X * M) \circ (Y * M) & \xleftarrow{(1 * \mu) \circ 1} & (X * (M * M)) \circ (Y * M). \end{array} \quad (7.8)$$

Recall from Section 5 that, when \mathcal{F} is horizontally left closed, not only does it become an \mathcal{F}_h -category, it becomes a pseudomonoid in $\mathcal{F}_h\text{-Cat}$ using the tensor $\hat{\circ}$. That is, $(\mathcal{F}, \hat{\circ}, \lceil \mathbf{1} \rceil)$ is a monoidal \mathcal{F}_h -category.

We are interested in when $(\mathcal{F}, \hat{\circ}, \lceil \mathbf{1} \rceil)$ is closed and when the closed structure lifts to \mathcal{F}^{*M} for a bimonoid M in \mathcal{F} .

Proposition 22 *The monoidal \mathcal{F}_h -category $(\mathcal{F}, \hat{\circ}, \lceil \mathbf{1} \rceil)$ is closed if and only if*

- (i) \mathcal{F}_v is a closed monoidal \mathcal{V} -category, and
- (ii) there exist \mathcal{V} -natural isomorphisms

$$X \circ (W * Y) \cong W * (X \circ Y) \cong (W * X) \circ Y.$$

Proof To say $(\mathcal{F}, \hat{\circ}, \lceil \mathbf{1} \rceil)$ is left closed is to say we have a “left hom” $\ell om(X, Y)$ and an \mathcal{F}_h -natural isomorphism

$$[X \circ Y, Z] \cong [X, \ell om(Y, Z)].$$

By Yoneda, this amounts to a \mathcal{V} -natural isomorphism

$$\mathcal{F}(W, [X \circ Y, Z]) \cong \mathcal{F}(W, [X, \ell om(Y, Z)]).$$

Since $[\cdot, \cdot]$ is the horizontal left hom for \mathcal{F} , this amounts to

$$\mathcal{F}(W * (X \circ Y), Z) \cong \mathcal{F}(W * X, \ellom(Y, Z)). \quad (7.9)$$

Taking $W = J$, we obtain

$$\mathcal{F}(X \circ Y, Z) \cong \mathcal{F}(X, \ellom(Y, Z)),$$

showing that ℓom is a left hom for \mathcal{F}_v as a monoidal \mathcal{V} -category. So (i) is implied. Now we have this, we can rewrite (7.9) as

$$\mathcal{F}(W * (X \circ Y), Z) \cong \mathcal{F}((W * X) \circ Y, Z)$$

which, again by Yoneda, is equivalent to

$$W * (X \circ Y) \cong (W * X) \circ Y. \quad (7.10)$$

Similarly, to say $(\mathcal{F}, \hat{\circ}, \mathbf{1})$ is right closed means

$$[X \circ Y, Z] \cong [Y, rom(X, Z)],$$

which means

$$\mathcal{F}(W * (X \circ Y), Z) \cong \mathcal{F}(W * Y, rom(X, Z)).$$

Taking $W = J$, we see that rom is a right hom for \mathcal{F}_v , and this leads to

$$W * (X \circ Y) \cong X \circ (W * Y). \quad (7.11)$$

This completes the proof. \blacksquare

Remark Under the condition of Proposition 22, it follows that the \mathcal{F}_h -functors

$$- * X, \quad - \circ X, \quad X \circ - \quad : \mathcal{F}_h \longrightarrow \mathcal{F}_h$$

all preserve weighted colimits.

Proposition 23 *For any duoidal \mathcal{V} -category \mathcal{F} , condition (ii) of Proposition 22 is equivalent to*

(ii)' there exist \mathcal{V} -natural isomorphisms

$$X * (J \circ Y) \cong X \circ Y \cong Y * (X \circ J). \quad (7.12)$$

Proof (ii) \implies (ii)' The second isomorphism of (ii)' comes from the first isomorphism of (ii) with $Y = J$ and W replaced by Y . The first isomorphism of (ii)' comes from the second isomorphism of (ii) with $X = J$ and W replaced by X . (ii)' \implies (ii) Using (ii)', we have

$$\begin{aligned} X \circ (W * Y) &\cong (W * Y) * (X \circ J) \\ &\cong W * (Y * (X \circ J)) \\ &\cong W * (X \circ Y), \text{ and} \\ (W * X) \circ Y &\cong (W * X) * (J \circ Y) \\ &\cong W * (X * (J \circ Y)) \\ &\cong W * (X \circ Y). \quad \blacksquare \end{aligned}$$

Theorem 24 Suppose \mathcal{F} is a duoidal \mathcal{V} -category which is horizontally left closed, has equalizers, and satisfies condition (ii)' of Proposition 23. Suppose M is a bimonoid in \mathcal{F} and regard \mathcal{F}^{*M} as a monoidal \mathcal{F}_h -category as in Theorem 18. The following conditions are equivalent:

- (i) M is a (left, right) Hopf bimonoid;
- (ii) $- * M$ is a (left, right) Hopf opmonoidal monad on \mathcal{F}_h .

If \mathcal{F}_v is a closed monoidal \mathcal{V} -category then these conditions are also equivalent to

- (iii) \mathcal{F}^{*M} is (left, right) closed and $U_M : \mathcal{F}^{*M} \rightarrow \mathcal{F}_h$ is strong (left, right) closed.

Proof (ii) \iff (iii) under the extra condition on \mathcal{F}_v by [BLV] as extended by [CLS].

(ii) \implies (i) by taking $X = Y = J$ in (7.7), we see that $v_\ell(X, Y) = v_\ell$.

(i) \implies (ii) Proposition 23 (ii)' and associativity of $*$ yield the isomorphisms

$$X \circ (Y * J) \cong Y * (X \circ J),$$

$$(Y * J) \circ X \cong Y * (J \circ X), \text{ and}$$

$$(Y * J) * X \cong Y * (J * X),$$

showing that $X \circ -, - \circ X$ and $- * X$ preserve the canonical weighted colimit of Proposition 19 (since $\text{colim}(W, S) \cong W * S$ when $S : \mathcal{J} \rightarrow \mathcal{F}_h$).

Using Proposition 20, we see that $v_\ell(X, Y)$ is an \mathcal{F}_h -natural transformation, in the variables X and Y , between two \mathcal{F}_h -functors that preserve weighted colimits of the form

$$\text{colim}(Z, J) \cong Z * J \cong Z.$$

By Proposition 21, $v_\ell(X, Y)$ is invertible if $v_\ell(J, J) = v_\ell$ is. \blacksquare

Example Any braided closed monoidal \mathcal{V} -category \mathcal{F} , regarded as duoidal by taking both $*$ and \circ to be the monoidal structure given on \mathcal{F} , is an example satisfying the conditions of Proposition 22.

Remark One reading of Proposition 23 (ii)' is that, to know \circ we only need to know $*$ and either $J \circ -$ or $- \circ J$. Proposition 22 (ii) also yields

$$Y \circ (W * \mathbf{1}) \cong W * Y \cong (W * \mathbf{1}) \circ Y \quad (7.13)$$

showing that to know $*$ we only need to know \circ and $- * \mathbf{1}$. From (7.12) we deduce

$$\mathbf{1} * (J \circ X) \cong X \cong \mathbf{1} * (X \circ J) \quad (7.14)$$

and from (7.13) we deduce

$$J \circ (X * \mathbf{1}) \cong X \cong (X * \mathbf{1}) \circ J \quad (7.15)$$

showing each of the composites

$$\begin{aligned} \mathcal{F} &\xrightarrow{-\circ J} \mathcal{F} \xrightarrow{1*-} \mathcal{F}, & \mathcal{F} &\xrightarrow{J\circ-} \mathcal{F} \xrightarrow{1*-} \mathcal{F}, \\ \mathcal{F} &\xrightarrow{-*1} \mathcal{F} \xrightarrow{J\circ-} \mathcal{F}, & \mathcal{F} &\xrightarrow{-*1} \mathcal{F} \xrightarrow{-\circ J} \mathcal{F} \end{aligned} \quad (7.16)$$

to be isomorphic to the identity \mathcal{V} -functor of \mathcal{F} . From the first and last of these we see that $-\circ J$ is an equivalence and

$$1 * - \cong - * 1 \quad (7.17)$$

both sides being inverse equivalences for $-\circ J$. From the second of (7.16) it then follows that $1 * -$ is an inverse equivalence for $J\circ-$. Consequently

$$J\circ- \cong -\circ J. \quad (7.18)$$

8 Warped monoidal structures

Let $\mathcal{A} = (\mathcal{A}, \otimes, I)$ be a monoidal category. The considerations at the end of Section 7 suggest the possibility of defining a tensor product on \mathcal{A} of the form

$$A \square B = TA \otimes B$$

for some suitable functor $T : \mathcal{A} \longrightarrow \mathcal{A}$. In the case of Section 7, the functor T was actually an equivalence but we will not assume that here in the first instance.

A *warping* of \mathcal{A} consists of the following data:

- (a) a functor $T : \mathcal{A} \longrightarrow \mathcal{A}$;
- (b) an object K of \mathcal{A} ;
- (c) a natural isomorphism

$$v_{a,b} : T(TA \otimes B) \longrightarrow TA \otimes TB ;$$

- (d) an isomorphism

$$v_0 : TK \longrightarrow I ; \text{ and}$$

- (e) a natural isomorphism

$$k_A : TA \otimes K \longrightarrow A ;$$

such that the following diagrams commute.

$$\begin{array}{ccc}
T(TA \otimes B) \otimes TC & \xrightarrow{v_{a,b} \otimes 1} & (TA \otimes TB) \otimes TC \quad (8.1) \\
\uparrow v_{TA \otimes B, C} & & \downarrow a_{TA, TB, TC} \\
T(T(TA \otimes B) \otimes C) & & TA \otimes (TB \otimes TC) \\
\downarrow T(v_{a,b} \otimes 1) & & \uparrow 1 \otimes v_{b,c} \\
T((TA \otimes TB) \otimes C) & & TA \otimes T(TB \otimes C) \\
\searrow Ta_{TA, TB, C} & & \nearrow v_{A, TB \otimes C} \\
& T(TA \otimes (TB \otimes C)) &
\end{array}$$

$$\begin{array}{ccc}
T(TA \otimes K) & \xrightarrow{v} & TA \otimes TK \quad (8.2) \\
\downarrow Tk_A & & \downarrow 1 \otimes v_0 \\
TA & \xleftarrow{r_{TA}} & TA \otimes I
\end{array}$$

Remark If $T : \mathcal{A} \rightarrow \mathcal{A}$ is essentially surjective on objects and fully-faithful on isomorphisms then all we need to build it up to a warping is $v_{A,B}$ as in (c) satisfying (8.1). For K and v_0 exist by essential surjectivity and k_A is defined by (8.2).

Proposition 25 *A warping of \mathcal{A} determines a monoidal structure on \mathcal{A} defined by the tensor product*

$$A \square B = TA \otimes B$$

with unit object K and coherence isomorphisms

$$\alpha : T(TA \otimes B) \otimes C \xrightarrow{v \otimes 1} (TA \otimes TB) \otimes C \xrightarrow{a} TA \otimes (TB \otimes C)$$

$$\ell : TK \xrightarrow{v_0 \otimes 1} I \otimes B \xrightarrow{\ell} B$$

$$r : TA \otimes K \xrightarrow{k} A.$$

Proof The pentagon condition for \square is obtained from (8.1) by applying $- \otimes D$. Similarly, the unit triangle is obtained from (8.2) by applying $- \otimes B$. \blacksquare

In investigating when \otimes and \square together formed a duoidal structure on \mathcal{A} , we realized we could use a lifting of Proposition 25 to a monoidal bicategory \mathcal{M} . We now describe this lifted version. The duoidal structure formed by \otimes and \square will be explained in an example.

A warping of a monoidal $A = (A, m, i)$ in a monoidal bicategory \mathcal{M} consists of

- (a) a morphism $t : A \rightarrow A$;
- (b) a morphism $k : I \rightarrow A$;

(c) an invertible 2-cell

$$\begin{array}{ccccc}
 & A \otimes A & \xrightarrow{m} & A & \\
 t \otimes 1 \nearrow & & \Downarrow v & & \searrow t \\
 A \otimes A & \xrightarrow{t \otimes t} & A \otimes A & \xrightarrow{m} & A ;
 \end{array}$$

(d) an invertible 2-cell

$$\begin{array}{ccc}
 & A & \\
 k \nearrow & \Downarrow v_0 & \searrow t \\
 I & \xrightarrow{i} & A ;
 \end{array}$$

(e) an invertible 2-cell

$$\begin{array}{ccc}
 & A & \\
 t \otimes k \nearrow & \Downarrow \kappa & \searrow m \\
 I & \xrightarrow{1} & A ;
 \end{array}$$

satisfying

$$\begin{array}{c}
 \begin{array}{ccccccc}
 & & A^{\otimes 3} & \xrightarrow{m \otimes 1} & A^{\otimes 2} & & \\
 & t \otimes 1 \otimes 1 \nearrow & & \Downarrow v \otimes 1 & & t \otimes 1 \searrow & \\
 A^{\otimes 3} & \xrightarrow{t \otimes t \otimes 1} & A^{\otimes 3} & \xrightarrow{m \otimes 1} & A^{\otimes 2} \cong & t \otimes t & A^{\otimes 2} \\
 & \searrow t \otimes t \otimes t & \downarrow 1 \otimes t \otimes t \cong & & \downarrow 1 \otimes t & & \\
 & & A^{\otimes 3} & \xrightarrow{m \otimes 1} & A^{\otimes 2} & \xleftarrow{v} & A \\
 & & \downarrow 1 \otimes m & \cong \alpha & \downarrow m & & \downarrow m \\
 & & A^{\otimes 2} & \xrightarrow{m} & A & \xleftarrow{t} & A
 \end{array} \\
 = & & & & & &
 \end{array} \tag{8.3}$$

$$\begin{array}{ccccc}
A^{\otimes 3} & \xrightarrow{m \otimes 1} & A^{\otimes 2} & \xrightarrow{t \otimes 1} & A^{\otimes 2} \\
& & \searrow m \otimes 1 & \nearrow \cong \alpha & \searrow m \\
& & A^{\otimes 3} & \xrightarrow{1 \otimes m} & A^{\otimes 2} & \xrightarrow{m} & A \\
& & \nearrow t \otimes t \otimes 1 & \nearrow t \otimes 1 & \nearrow v & \searrow t \\
& & A^{\otimes 3} & \xrightarrow{1 \otimes t \otimes 1} & A^{\otimes 3} & \xrightarrow{1 \otimes m} & A^{\otimes 2} \\
& & \searrow t \otimes 1 \otimes 1 & \searrow t \otimes 1 \otimes 1 & \searrow t \otimes t & \nearrow m \\
& & A^{\otimes 3} & \xrightarrow{1 \otimes t \otimes 1} & A^{\otimes 3} & \xrightarrow{1 \otimes m} & A^{\otimes 2} & \xrightarrow{1 \otimes t} & A^{\otimes 2} \\
& & \searrow 1 \otimes t \otimes t & \searrow 1 \otimes v & \searrow 1 \otimes m & \nearrow t \otimes t & \nearrow m \\
& & A^{\otimes 3} & & A^{\otimes 3} & & A^{\otimes 2} & & A^{\otimes 2}
\end{array}$$

Diagram illustrating a complex commutative structure involving tensor products of \$A\$ and maps \$m, t, v, \alpha, \rho, i, v_0, k\$. The diagram shows multiple paths between \$A^{\otimes 3}\$ and \$A^{\otimes 2}\$, with various isomorphisms and natural transformations labeled.

and

$$\begin{array}{ccccc}
A & \xrightarrow{1 \otimes k} & A^{\otimes 2} & \xrightarrow{t \otimes 1} & A^{\otimes 2} \\
& & \searrow t \otimes t & \nearrow v & \searrow m \\
& & A^{\otimes 2} & \xrightarrow{1 \otimes t} & A^{\otimes 2} \\
& & \searrow 1 \otimes v_0 & \searrow 1 \otimes i & \searrow m \\
& & A & \xrightarrow{1 \otimes k} & A^{\otimes 2} \\
& & \searrow 1 & \searrow \rho & \searrow t \\
& & A & & A
\end{array}$$

Diagram illustrating a commutative structure involving tensor products of \$A\$ and maps \$m, t, v, \rho, i, v_0, k\$. The diagram shows paths between \$A\$ and \$A^{\otimes 2}\$, with various isomorphisms and natural transformations labeled.

=

$$\begin{array}{ccccc}
& & A^{\otimes 2} & & \\
& \nearrow t \otimes k & \Downarrow \kappa & \nwarrow m & \\
A & \xrightarrow{\quad 1 \quad} & A & & A \\
\downarrow t & & \cong & & \downarrow t \\
A & \xrightarrow{\quad 1 \quad} & A & & A
\end{array}$$

Proposition 26 *A warping of a monoidale A determines a monoidale structure on A defined by*

$${}_t m : A \otimes A \xrightarrow{t \otimes 1} A \otimes A \xrightarrow{m} A$$

$$I \xrightarrow{k} A$$

$$\begin{array}{ccccccc}
A^{\otimes 3} & \xrightarrow{t \otimes 1 \otimes 1} & A^{\otimes 3} & \xrightarrow{m \otimes 1} & A^{\otimes 2} & \xrightarrow{t \otimes 1} & A^{\otimes 2} \\
& \searrow t \otimes t \otimes 1 & & \Downarrow v \otimes 1 & \nearrow m \otimes 1 & & \\
& & A^{\otimes 3} & & & & \\
1 \otimes t \otimes 1 \downarrow & & \cong & & \cong \alpha & & m \downarrow \\
A^{\otimes 3} & \xrightarrow{1 \otimes m} & A^{\otimes 2} & \xrightarrow{t \otimes 1} & A^{\otimes 2} & \xrightarrow{m} & A
\end{array}$$

$$\begin{array}{ccccc}
& A^{\otimes 2} & \xrightarrow{t \otimes 1} & A^{\otimes 2} & \\
& \nearrow k \otimes 1 & \Downarrow v_0 \otimes 1 & \nwarrow m & \\
A & \xrightarrow{\quad i \otimes 1 \quad} & A & & A \\
& \searrow 1 & & \cong \lambda & \\
& & & &
\end{array}$$

$$\begin{array}{ccccc}
A & \xrightarrow{t \otimes k} & A^{\otimes 2} & \xrightarrow{m} & A \\
& \searrow 1 & \Downarrow \kappa & \nearrow & \\
& & & &
\end{array}$$

Proof Conditions (8.3) and (8.4) yield the two axioms for a monoidale $(A, {}_t m, k)$. ■

Example Suppose \mathcal{F} is a duoidal \mathcal{V} -category satisfying the second isomorphism of (7.13). Define a \mathcal{V} -functor $T : \mathcal{F} \rightarrow \mathcal{F}$ by

$$T = - * \mathbf{1}.$$

The horizontal right unit isomorphism gives

$$T(J) = J * \mathbf{1} \cong \mathbf{1}$$

and (7.13) gives

$$\begin{aligned} T(TA \circ B) &= ((A * \mathbf{1}) \circ B) * \mathbf{1} \\ &\cong (A * B) * \mathbf{1} \\ &\cong A * (B * \mathbf{1}) \\ &\cong (A * \mathbf{1}) \circ (B * \mathbf{1}) \\ &= TA \circ TB. \end{aligned}$$

Finally, we have

$$\begin{aligned} TA \circ J &= (A * \mathbf{1}) \circ J \\ &\cong A * J \\ &\cong A. \end{aligned}$$

This gives an example of a warping in $\mathcal{M} = \mathcal{V}\text{-Cat}$ of the monoidale (= monoidal \mathcal{V} -category) \mathcal{F}_v . Proposition 26 gives back \mathcal{F}_h .

Example Consider the case of $\mathcal{M} = \text{Mon}(\mathcal{V}\text{-Cat})$. A monoidale is a duoidal \mathcal{V} -category $(\mathcal{F}_h, \circ, \mathbf{1})$. A warping of this monoidale consists of a monoidal \mathcal{V} -functor $T : \mathcal{F}_h \rightarrow \mathcal{F}_h$, a monoid K in \mathcal{F}_h , a horizontally monoidal \mathcal{V} -natural isomorphism $v : T(TA \circ B) \cong TA \circ TB$, a horizontal monoid isomorphism $v_0 : TK \cong \mathbf{1}$, and a horizontally monoidal \mathcal{V} -natural isomorphism $k : TA \circ K \cong A$, subject to the two conditions. Proposition 26 gives the recipe for obtaining a duoidal \mathcal{V} -category $(\mathcal{F}_h, (T-) \circ -, K)$. In particular, take $\mathcal{V} = \text{Set}$ and consider a lax braided monoidal category $\mathcal{A} = (\mathcal{A}, \otimes, I, c)$ as a duoidal category; the lax braiding gives the monoidal structure on $\otimes : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$. A warping consists of a monoidal functor $T : \mathcal{A} \rightarrow \mathcal{A}$, a monoid K in \mathcal{A} , a monoidal natural $v : T(TA \otimes B) \cong TA \otimes TB$, a monoid isomorphism $v_0 : TK \cong I$, and a monoidal natural $k : TA \otimes K \cong A$, satisfying the conditions (8.1) and (8.2). Proposition 26 then shows that the recipe of Proposition 25 yields a duoidal category $(\mathcal{A}, \otimes, I, \square, K)$.

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